CPSC 536N Randomized Algorithms (Winter 2014-15, Term 2)
Assignment 4

Due: Wednesday April 1st, in class.

Question 1: Dependency Graphs

Give an example of a probability space and four events $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4$ such that

- there is no dependency graph with just three edges
- there are (at least) two different dependency graphs with four edges. (It is possible that these two graphs are isomorphic.)

You should prove those two properties.

Hint: How many ways do you know to make events (or random variables) that are not mutually independent?

Question 2: Nice LLL

The following theorem gives another way to remove the symmetry from the Symmetric LLL.

Theorem 1 (Nice LLL). Assume that $\Pr[\mathcal{E}_i] < 1/2$ for every i. Suppose that there is a dependency graph satisfying

$$\sum_{j \in \Gamma(i)} \Pr[\mathcal{E}_j] \;\leq\; 1/4 \qquad \forall i.$$

Then $\Pr\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}_i}\right] > 0$.

Use the General LLL stated in Lecture 18 to prove the Nice LLL.

Hint: First prove that

$$1 - \sum_{i=1}^{n} a_i \;\leq\; \prod_{i=1}^{n} (1 - a_i) \tag{1}$$

for any real numbers a_i satisfying $a_i \geq 0$ and $\sum_{i=1}^{n} a_i \leq 1$.

Question 3: Not-All-Equal SAT

One variant of the satisfiability problem is called Not-All-Equal SAT. An instance of this problem involves Boolean variables $x_1, ..., x_m$ and n clauses. Clause i involves some subset of the variables $\{ x_j : j \in S_i \}$, and it requires that these variables should not all take the *same* value. In other words, clause i equals the Boolean formula

$$\left(\bigcup_{j \in S_i} x_j \right) \cap \left(\bigcup_{j \in S_i} \overline{x_j} \right).$$

The size of clause i is defined to be $|S_i|$. The instance is satisfiable if all clauses can be simultaneously satisfied.

Consider an instance of Not-All-Equal SAT in which:

- Each clause has size at least 3.
- Each clause shares a variable with at most a_k clauses of size k, for each $k \geq 3$, where $\sum_k a_k 2^{-k} \leq 1/8$.

Prove that the instance is satisfiable.

Question 4: Revisiting Assignment 1, Question 4

Let M be a matrix with m rows, n columns such that

- every entry $M_{i,j} \in \{0, 1\}$,
- every row sums to r (i.e., $\sum_{j=1}^{n} M_{i,j} = r$ for all i),
- every column sums to c (i.e., $\sum_{i=1}^{m} M_{i,j} = c$ for all j.)

Show that there exists a vector $Y \in \{0, 1\}^n$ such that, letting $Z = M \cdot Y$, we have

$$\max_i Z_i \;\leq\; (r/2) + O(\sqrt{r \log(rc)})$$
$$\min_i Z_i \;\geq\; (r/2) - O(\sqrt{r \log(rc)}).$$