PROBABILISTIC GRAPHICAL MODELS
CPSC 532C (TOPICS IN AI)
STAT 521A (TOPICS IN MULTIVARIATE ANALYSIS)

LECTURE 8

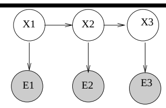
Kevin Murphy

Wednesday 6 October, 2004

ADMINISTRIVIA

- Next Monday: no class (thanksgiving)
- Next Wednesday: lecture by Brent Boerlage.

WHAT'S WRONG WITH VARIABLE ELIMINATION?



- Consider computing $P(X_i|y_{1:N})$ for each i using variable elimination. This would take $O(N^2)$ time.
- However, there is a lot of repeated computation.

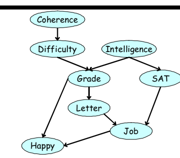
$$P(X_1|e_{1:3}) \propto P(X_1)p(e_1|X_1)\sum_{X_2} P(X_2|X_1)p(e_2|X_2)\sum_{X_3} P(X_3|X_2)p(e_3$$

$$P(X_2|e_{1:3}) \propto \sum_{X_1} P(X_1)p(e_1|X_1)P(X_2|X_1)p(e_2|X_2)\sum_{X_3} P(X_3|X_2)p(e_3|$$

$$P(X_3|e_{1:3}) \propto \sum_{X_1} P(X_1)p(e_1|X_1)\sum_{X_2} P(X_2|X_1)p(e_2|X_2)P(X_3|X_2)p(e_3|$$

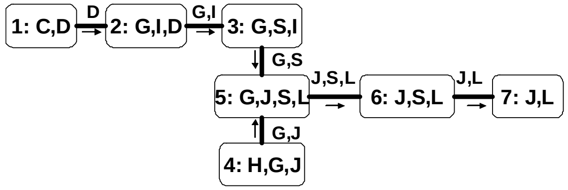
- We will show how to use caching to compute all N marginals in $O(N)$ time.

RECALL VARIABLE ELIMINATION



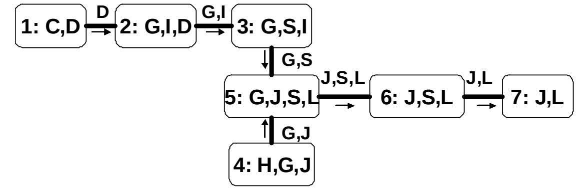
$$\begin{aligned}
P(J) &= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \phi_H(H,G,J) \sum_I \phi_S(S,I)\phi_I(I) \sum_D \phi_{(}(G,I,D) \sum_C \underbrace{\phi_C(C)\phi_D(D,C)}_{\psi_1(C,D)} \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \phi_H(H,G,J) \sum_I \phi_S(S,I)\phi_I(I) \sum_D \underbrace{\phi_{(}(G,I,D)\tau_1(D)}_{\psi_2(D,G,I)} \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \phi_H(H,G,J) \sum_I \underbrace{\phi_S(S,I)\phi_I(I)\tau_2(G,I)}_{\psi_3(I,G,S)} \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \underbrace{\phi_H(H,G,J)}_{\psi_4(H,G,J)} \tau_3(G,S) \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \underbrace{\phi_L(L,G)\tau_4(G,J)\tau_3(G,S)}_{\psi_5(G,J,L,S)} \\
&= \sum_L \sum_S \underbrace{\phi_J(J,L,S)\tau_5(J,L,S)}_{\psi_6(S,J,L)} \\
&= \sum_L \underbrace{\tau_6(J,L)}_{\psi_7(L,J)}
\end{aligned}$$

Cluster tree



$$
\begin{aligned}
P(J) &= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \phi_H(H,G,J) \sum_I \phi_S(S,I)\phi_I(I) \sum_D \phi_{(}G,I,D) \sum_C \underbrace{\phi_C(C)\phi_D(D,C)}_{\psi_1(C,D)} \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \phi_H(H,G,J) \sum_I \phi_S(S,I)\phi_I(I) \sum_D \underbrace{\phi_{(}G,I,D)\tau_1(D)}_{\psi_2(D,G,I)} \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \phi_H(H,G,J) \sum_I \underbrace{\phi_S(S,I)\phi_I(I)\tau_2(G,I)}_{\psi_3(I,G,S)} \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \phi_L(L,G) \sum_H \underbrace{\phi_H(H,G,J)}_{\psi_4(H,G,J)} \tau_3(G,S) \\
&= \sum_L \sum_S \phi_J(J,L,S) \sum_G \underbrace{\phi_L(L,G)\tau_4(G,J)\tau_3(G,S)}_{\psi_5(G,J,L,S)} \\
&= \sum_L \sum_S \underbrace{\phi_J(J,L,S)\tau_5(J,L,S)}_{\psi_6(S,J,L)} \\
&= \sum_L \underbrace{\tau_6(J,L)}_{\psi_7(L,J)}
\end{aligned}
$$

Junction trees

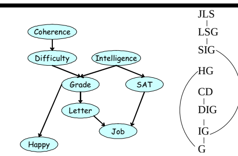


- A cluster graph is called a **junction tree** if it is a tree and if for every $X \in C_i \cap C_j$, then X occurs in every cluster in the (unique) path between C_i and C_j. (The book incorrectly calls this the **running intersection property**.)
- Thm 8.1.5: Variable elimination produces a junction tree.
- Pf: once a variable is encountered in the ordering, it occurs in all factors that mention it until it is summed out. Once it has been removed, it cannot be used again.

Constructing an elimination tree

- The clusters (nodes) produced by variable elimination using order $\prec$ applied to G are (non-maximal) cliques in the induced graph $I_{G,\prec}$.
- These clusters C_i are called **elimination sets**.
- We can connect the esets into a tree that satisfies the jtree property in 2 steps:
  1. Run the variable elimination algorithm. Let v_i be the variable eliminated at the i'th step, and C_i be the set of variables in v_i's bucket at that time (so $\tau_i = \sum_{v_i} \psi_i(C_i)$).
  2. Connect $C_i - C_j$ if τ_i goes into j's bucket, i.e., j is the largest index of a vertex in $C_i \setminus \{v_i\}$.
- The etree has the property that residuals $R_i = C_i \setminus S_{ij}$ are singleton sets, where $S_{ij} = C_i \cap C_j$ is the separator between S_i and S_j.

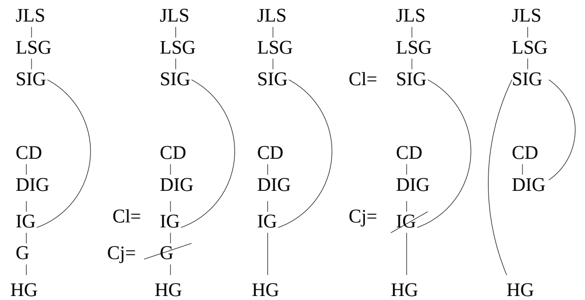
Example of etree construction



$$
\begin{aligned}
P(e) &= \sum_G \sum_I \phi_I(I) \sum_D \phi_G(G,I,D) \sum_C \phi_D(D,C)\phi_C(C) \sum_H \phi_H(H,G) \sum_S \phi_S(S,I) \sum_L \phi_L(L,G) \underbrace{\sum_J \phi_J(J,L,S)}_{\tau_1(L,S)} \\
&= \sum_G \sum_I \phi_I(I) \sum_D \phi_G(G,I,D) \sum_C \phi_D(D,C)\phi_C(C) \sum_H \phi_H(H,G) \sum_S \phi_S(S,I) \underbrace{\sum_L \phi_L(L,G)\tau_1(L,S)}_{\tau_2(G,S)} \\
&= \sum_G \sum_I \phi_I(I) \sum_D \phi_G(G,I,D) \sum_C \phi_D(D,C)\phi_C(C) \sum_H \phi_H(H,G) \underbrace{\sum_S \phi_S(S,I)\tau_2(G,S)}_{\tau_3(G,I)} \\
&= \sum_G \sum_I \phi_I(I)\tau_3(G,I) \sum_D \phi_G(G,I,D) \sum_C \phi_D(D,C)\phi_C(C) \underbrace{\sum_H \phi_H(H,G)}_{\tau_4(G)} \\
&= \sum_G \tau_4(G) \sum_I \phi_I(I)\tau_3(G,I) \sum_D \phi_G(G,I,D) \underbrace{\sum_C \phi_D(D,C)\phi_C(C)}_{\tau_5(D)} \\
&= \sum_G \tau_4(G) \sum_I \phi_I(I)\tau_3(G,I) \underbrace{\sum_D \phi_G(G,I,D)\tau_5(D)}_{\tau_6(G,I)} \\
&= \sum_G \tau_4(G) \underbrace{\sum_I \phi_I(I)\tau_3(G,I)\tau_6(G,I)}_{\tau_7(G)}
\end{aligned}
$$

From etree to jtree of maximal cliques

- Thm 8.4.1: We can remove non-maximal cliques and preserve the jtree property as follows.
- Let C_j, C_i be a pair of cliques s.t. $C_j \subset C_i$. By the jtree property, C_j is a subset of all cliques on the path from C_j to C_i.
- Let C_l be a neighbor of C_j st $C_j \subseteq C_l$. We remove C_j and connect all of its neighbors to C_l.

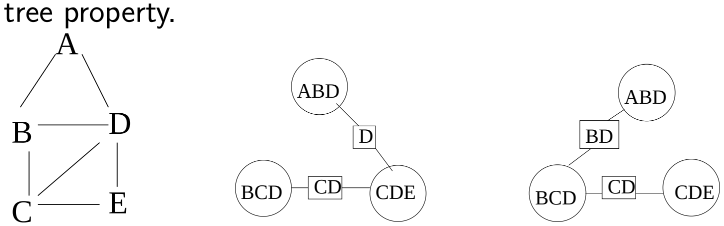


From chordal graph to jtree of maximal cliques

- Thm 8.4.1 shows that there is a jtree for F whose cliques are the maximal cliques in $I_{F,\prec}$.
- Suppose we are given the chordal graph $I_{F,\prec}$; how can we find the jtree directly?
- Step 1: find the maximal cliques of the chordal graph.
  - Finding maximal cliques is in general NP-hard.
  - But for chordal graphs, we can just run max cardinality search (or some other elimination algorithm) and save the maximal cliques.
- Step 2: connect the cliques so as to satisfy the jtree property.

Junction tree property

- Not every clique tree derived from a triangulated graph has the junction tree property.



- Defn: the weight of a clique tree is

$$W(T) = \sum_{j=1}^{M-1} |S_j|$$

where M is the number of cliques and S_j are separators.

- So the left graph (that does not have the jtree property) has weight $|\{C, D\}| + |\{D\}| = 3$, whereas the right graph (that does have the jtree property) has weight $|\{C, D\}| + |\{B, D\}| = 4$,

Jtree iff MWST

- Thm: a clique tree is a junction tree iff it is a maximal weight spanning tree.
- Proof. For a tree, the number of times X_k appears in all separators is one less than the number of times X_k appears in all cliques:

$$\sum_{j=1}^{M-1} 1(X_k \in S_k) \leq \sum_{i=1}^{M} 1(X_k \in C_i) - 1$$

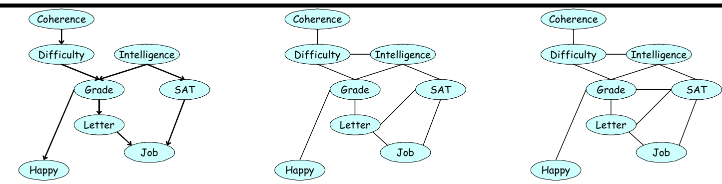
which becomes an inequality if the subgraph induced by X_k is a tree (i.e., T is a jtree).

Jtree iff MWST

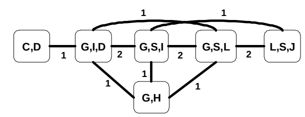
$$
\begin{aligned}
w(T) &= \sum_{j=1}^{M-1} |S_j| \\
&= \sum_{j=1}^{M-1} \sum_{k=1}^{N} 1(X_k \in S_j) \\
&= \sum_{k=1}^{N} \sum_{j=1}^{M-1} 1(X_k \in S_j) \\
&\leq \sum_{k=1}^{N} \left[\sum_{i=1}^{M} 1(X_k \in C_i) - 1 \right] \\
&= \sum_{i=1}^{M} \sum_{k=1}^{N} 1(X_k \in C_i) - N \\
&= \sum_{i=1}^{M} |C_i| - N
\end{aligned}
$$

- This is an equality iff T is a jtree.
- To make a jtree from a set of cliques of a chordal graph
  - Build a junction graph, where weight on edge $C_i - C_j$ is $|S_{ij}|$.
  - Find MWST using Prim's or Kruskal's algorithm.

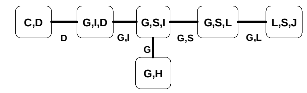
From Bayes net to jtree



BN Moralize Triangulate

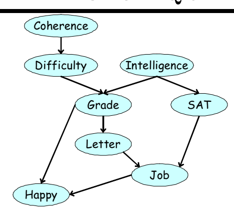


Jgraph



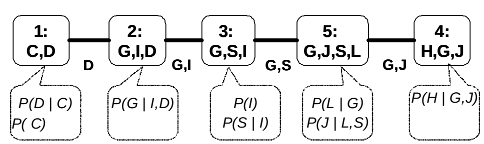
Jtree

Initializing clique trees



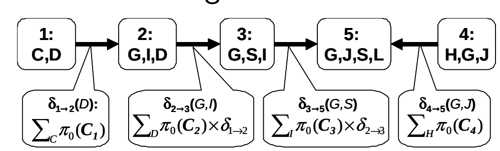
- The potential for clique c is initialized to the product of all assigned factors from the model:

$$\pi_j(C_j) = \prod_{\phi:\alpha(\phi)=j} \phi$$



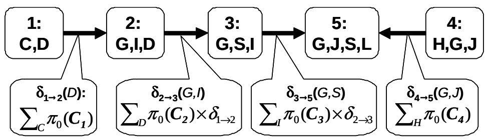
Message passing in clique trees

- To compute $P(J)$, we find some clique that contains J (eg. C_5) and call it the root.
- We then send messages from the leaves up to the root.
- A node C_i can send to C_j (closer to the root) once it has received messages from all its other neighbors C_k.
- The order to send the messages is called a schedule.



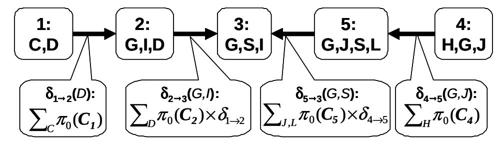
Collect to C_5

$$
\begin{aligned}
\delta_{1\to 2}(D) &= \sum_C \pi_1^0(C) \\
\pi_2(G,I,D) &= \pi_2^0(G,I,D)\delta_{1\to 2}(D) \\
\delta_{2\to 3}(G,I) &= \sum_D \pi_2(G,I,D) \\
\pi_3(G,S,I) &= \pi_3^0(G,S,I)\delta_{2\to 3}(G,I) \\
\delta_{3\to 5}(G,S) &= \sum_I \pi_3(G,S,I) \\
\delta_{4\to 5}(G,J) &= \sum_H \pi_4^0(H,G,J) \\
\pi_5(G,J,S,L) &= \pi_5^0(G,J,S,L)\delta_{3\to 5}(G,S)\delta_{4\to 5}(G,J)
\end{aligned}
$$



Collect to C_3

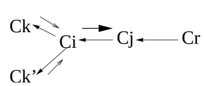
$$
\begin{aligned}
\delta_{1\to 2}(D) &= \sum_C \pi_1^0(C) \\
\pi_2(G,I,D) &= \pi_2^0(G,I,D)\delta_{1\to 2}(D) \\
\delta_{2\to 3}(G,I) &= \sum_D \pi_2(G,I,D) \\
\delta_{4\to 5}(G,J) &= \sum_H \pi_4^0(H,G,J) \\
\pi_5(G,J,S,L) &= \pi_5^0(G,J,S,L)\delta_{4\to 5}(G,J) \\
\delta_{5\to 3}(G,S) &= \sum_{J,L} \pi_5(G,J,S,L) \\
\pi_3(G,S,I) &= \pi_3^0(G,S,I)\delta_{2\to 3}(G,I)\delta_{5\to 3}(G,S)
\end{aligned}
$$



General procedure for upwards pass

$\psi_r^1 \stackrel{\text{def}}{=}$ function Ctree-VE-up$(\{\phi\}, T, \alpha, r)$

$DT :=$ mkRootedTree(T, r)
$\{\psi_i^0\} :=$ initializeCliques(ϕ, α)
for $i \in$ postorder(DT)
  $j := pa(DT, i)$
  $\delta_{i\to j} :=$ VE-msg$(\{\delta_{k\to i} : k \in ch(DT, i)\}, \psi_i^0)$
end
$\psi_r^1 := \psi_r^0 \prod_{k \in ch(DT,r)} \delta_{k\to r}$



Sub-functions

$\{\psi_i^0\} \stackrel{\text{def}}{=}$ function initializeCliques(ϕ, α)

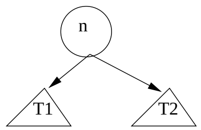
for $i := 1 : C$
  $\psi_i^0(C_i) = \prod_{\phi:\alpha(\phi)=i} \phi$

$\delta_{i\to j} \stackrel{\text{def}}{=}$ function VE-msg$(\{\delta_{k\to i}\}, \psi_i^0)$

$\psi_i^1(C_i) := \psi_i^0(C_i) \prod_k \delta_{k\to i}$
$\delta_{i\to j}(S_{i,j}) := \sum_{C_i \setminus S_{ij}} \psi_i^1(C_i)$

Tree traversal orders

```
preorder =  [n, pre(T1), pre(T2)]  (parents then children)
inorder =   [in(T1), n, in(T2)]
postorder = [post(T1), post(T2), n]  (children then parents
```



Depth first search of a graph

- See e.g., "Introduction to algorithms", Cormen, Leiserson, Rivest
- Initialize all nodes white; when first discovered, paint gray; when finished (all neighbors explored), paint black.
- $d(u)$ = discovery time, $f(u)$ = finish time, $\pi(u)$ = predecessor in the dfs ordering

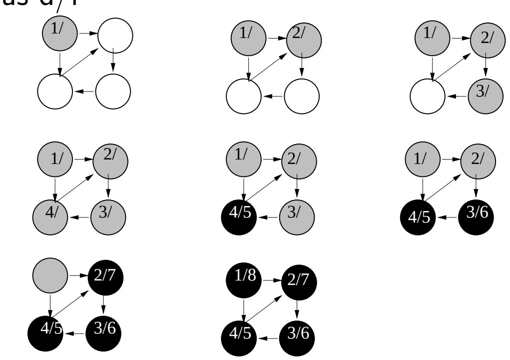
```
(d, f, pi) = function dfs(G)
for each vertex u
  color(u) := white
  pi(u) := []
time := 0
for each u
  if color(u)==white
  then dfs-visit(u)
```

Depth first search of a graph

```
function dfs-visit(u)
color(u) := gray
d(u) := (time := time + 1)
for each v in neighbors(u)
  if color(v) == white
  then pi(v) := u;
       dfs-visit(v)
  elseif color(v) == gray
  then  cycle detected
color(u) := black
f(u) := (time := time + 1)
```

Depth first search of a graph

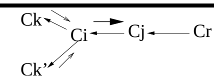
Nodes labeled as d/f



Uses of DFS

- For message passing on an undirected tree:
  - We can root a tree at R and make all arcs point away from R by starting the DFS at R and connecting $\pi(i) \rightarrow i$.
  - preorder (parents then children) = nodes sorted by discovery time
  - postorder (children then parents) = nodes sorted by finish time
- For visiting nodes in a DAG in a topological order (parents before children)
  - Topological order = nodes sorted by *reverse* finish time
- For checking if a DAG has cycles
  - Run DFS, see if you ever encounter a back-edge to a gray node
- For finding strongly connected components

Correctness of upwards pass



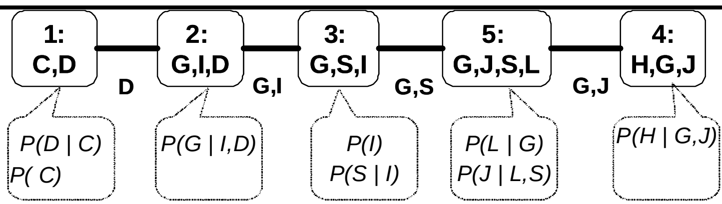
- Consider edge $C_i - C_j$ in the clique tree. Let $F_{\prec(i \rightarrow j)}$ be all factors on the C_i side, and $V_{\prec(i \rightarrow j)}$ be all variables on the C_i side that are not in S_{ij}.
- Thm 8.2.3: the message from i to j summarizes everything to the left of the edge (since S_{ij} separates the left from the right):

$$\delta_{i \rightarrow j}(S_{ij}) = \sum_{V_{\prec(i \rightarrow j)}} \prod_{\phi \in F_{\prec(i \rightarrow j)}} \phi$$

- Corollary 8.2.4: for the root clique,

$$\pi_r(C_r) = \sum_{X \setminus C_r} P'(X)$$

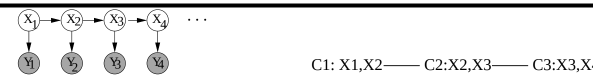
Meaning of the messages



- e.g., for edge $C_3 - C_5$,

$$\begin{aligned} F_{\prec(3 \rightarrow 5)} &= \{P(D|C), P(C), P(G|I,D), P(I), P(S|I)\} \\ V_{\prec(3 \rightarrow 5)} &= \{C, D, I\} \\ \delta_{3 \rightarrow 5}(G, S) &= \sum_{C,D,I} P(D|C)P(C)P(G|I,D)P(I)P(S|I) \end{aligned}$$

Meaning of the messages



- Partial messages may not be probability distributions unless the ordering is topologically consistent with a Bayes net.
- Causal order

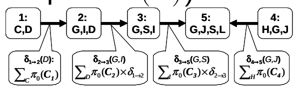
$$\begin{aligned} \delta_{1 \rightarrow 2}(X_2) &= \sum_{X_1} P(X_1)p(y_1|X_1)P(X_2|X_1)p(y_2|X_2) \propto P(X_2|y_{1:2}) \\ \delta_{2 \rightarrow 3}(X_3) &= \sum_{X_2} \delta_{1 \rightarrow 2}(X_2)P(X_3|X_2)p(y_3|X_3) \propto P(X_3|y_{1:3}) \end{aligned}$$

- Anti-causal order

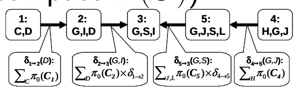
$$\begin{aligned} \delta_{3 \rightarrow 2}(X_3) &= \sum_{X_4} P(X_4|X_3)p(y_4|X_4) = p(y_4|X_3) \\ \delta_{2 \rightarrow 1}(X_2) &= \sum_{X_3} \delta_{3 \rightarrow 2}(X_2)P(X_3|X_2)p(y_3|X_3) = p(y_{3:4}|X_3) \end{aligned}$$

Computing messages for each edge

- If we collect to C_5 (to compute $P(J)$)



- If we collect to C_3 (to compute $P(G)$)

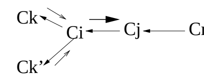


- The messages $\delta_{1\rightarrow 2}$, $\delta_{2\rightarrow 3}$, $\delta_{4\rightarrow 5}$ are the same in both cases.
- In general, if the root R is on the C_j side, the message from $C_i \rightarrow C_j$ is independent of R. If the root is on the C_i side, the message from $C_j \rightarrow C_i$ is independent of R.
- Hence we can send an edge along each edge in both directions and thereby compute all marginals in $O(C)$ time.

Shafer-Shenoy algorithm

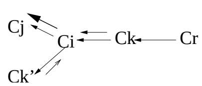
$\{\psi_i^1\} \stackrel{\text{def}}{=}$ function Ctree-VE-calibrate$(\{\phi\}, T, \alpha)$

$R :=$ pickRoot(T)
$DT :=$ mkRootedTree(T, R)
$\{\psi_i^0\} :=$ initializeCliques(ϕ, α)
(* Upwards pass *)
for $i \in$ postorder(DT)
    $j := pa(DT, i)$
    $\delta_{i\rightarrow j} :=$ VE-msg$(\{\delta_{k\rightarrow i} : k \in ch(DT, i)\}, \psi_i^0)$



Shafer-Shenoy algorithm

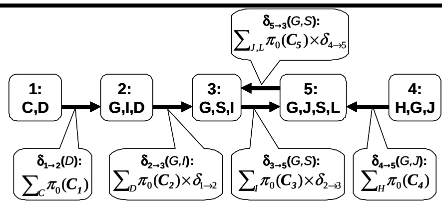
(* Downwards pass *)
for $i \in$ preorder(DT)
    for $j \in ch(DT, i)$
        $\delta_{i\rightarrow j} =$ VE-msg$(\{\delta_{k\rightarrow i} : k \in N_i \setminus j\}, \psi_i^0)$
(* Combine *)
for $i := 1 : C$
    $\psi_i^1 := \psi_i^0 \prod_{k \in N_i} \delta_{k\rightarrow i}$

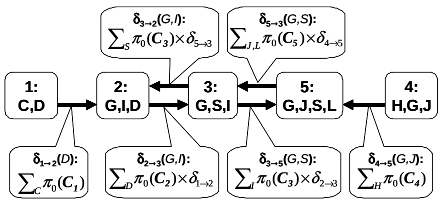


Correctness of Shafer Shenoy

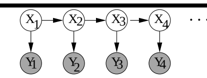
- Thm 8.2.7: After running the algorithm,
$$\psi_i^1(C_i) = \sum_{X \setminus C_i} P'(X, e)$$
- Pf: the incoming messages $\delta_{k\rightarrow i}$ are exactly the same as those computed by making C_i be the root; so correctness follows from the correctness of collect-to-root (upwards pass).
- The posterior of any set of nodes contained in a clique can be computed using
$$P(C_i|e) = \psi_i^1(C_i)/p(e)$$
where the likelihood of the evidence can be computed from any clique
$$p(e) = \sum_{c_i} \psi_i^1(c_i)$$

Example of distributing from root C_5





Shafer Shenoy for HMMs



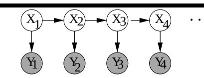
C1: X1,X2—— C2:X2,X3—— C3:X3,X4

$$
\begin{aligned}
\psi_t^0(X_t, X_{t+1}) &= P(X_{t+1}|X_t)p(y_{t+1}|X_{t+1}) \\
\delta_{t\to t+1}(X_{t+1}) &= \sum_{X_t} \delta_{t-1\to t}(X_t)\psi_t^0(X_t, X_{t+1}) \\
\delta_{t\to t-1}(X_t) &= \sum_{X_{t+1}} \delta_{t+1\to t}(X_{t+1})\psi_t^0(X_t, X_{t+1}) \\
\psi_t^1(X_t, X_{t+1}) &= \delta_{t-1\to t}(X_t)\delta_{t+1\to t}(X_{t+1})\psi_t^0(X_t, X_{t+1})
\end{aligned}
$$

Forwards-backwards algorithm for HMMs

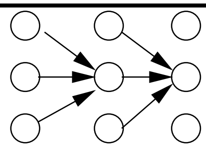
$$
\begin{aligned}
\alpha_t(i) &\stackrel{\text{def}}{=} \delta_{t-1\to t}(i) = P(X_t = i, y_{1:t}) \\
\beta_t(i) &\stackrel{\text{def}}{=} \delta_{t\to t-1}(i) = p(y_{t+1:T}|X_t = i) \\
\xi_t(i,j) &\stackrel{\text{def}}{=} \psi_t^1(X_t = i, X_{t+1} = j) = P(X_t = i, X_{t+1} = j, y_{1:T} \\
P(X_{t+1} = j|X_t = i) &\stackrel{\text{def}}{=} A(i,j) \\
p(y_t|X_t = i) &\stackrel{\text{def}}{=} B_t(i) \\
\alpha_t(j) &= \sum_i \alpha_{t-1}(i)A(i,j)B_t(j) \\
\beta_t(i) &= \sum_j \beta_{t+1}(j)A(i,j)B_{t+1}(j) \\
\xi_t(i,j) &= \alpha_t(i)\beta_{t+1}(j)A(i,j)B_{t+1}(j) \\
\gamma_t(i) &\stackrel{\text{def}}{=} P(X_t = i|y_{1:T}) \propto \alpha_t(i)\beta_t(j) \propto \sum_j \xi_t(i,j)
\end{aligned}
$$

Forwards-backwards algorithm, matrix-vector form



$$
\begin{aligned}
\alpha_t(j) &= \sum_i \alpha_{t-1}(i)A(i,j)B_t(j) \\
\alpha_t &= (A^T\alpha_{t-1}). * B_t \\
\beta_t(i) &= \sum_j \beta_{t+1}(j)A(i,j)B_{t+1}(j) \\
\beta_t &= A(\beta_{t+1}. * B_{t+1}) \\
\xi_t(i,j) &= \alpha_t(i)\beta_{t+1}(j)A(i,j)B_{t+1}(j) \\
\xi_t &= \left(\alpha_t(\beta_{t+1}. * B_{t+1})^T\right). * A \\
\gamma_t(i) &\propto \alpha_t(i)\beta_t(j) \\
\gamma_t &\propto \alpha_t. * \beta_t
\end{aligned}
$$

HMM trellis



- Forwards algorithm uses dynamic programming to efficiently sum over all possible paths that state i at time t.

$$
\begin{aligned}
\alpha_t(i) &\stackrel{\text{def}}{=} P(X_t = i, y_{1:t}) \\
&= \left[\sum_{X_1} \cdots \sum_{X_{t-1}} P(X_1, \ldots, X_t - 1, y_{1:t-1}) P(X_t | X_{t-1})\right] p(y_t | X_t) \\
&= \left[\sum_{X_{t-1}} P(X_t - 1, y_{1:t-1}) P(X_t | X_{t-1})\right] p(y_t | X_t) \\
&= \left[\sum_{X_{t-1}} \alpha_{t-1}(X_{t-1}) P(X_t | X_{t-1})\right] p(y_t | X_t)
\end{aligned}
$$