Homework on Hybrid & Continuous Systems

Keep your paper and pencil answers brief. For simulations, submit a **labeled** plot and your **documented** MATLAB files. In addition to the standard labels and documentation, each plot and MATLAB file should explicitly state your name and to which question they apply.

1. **Classifying Differential Equations.**

   (a) Let $\dot{x} = e^x$ with $x(0) = -\log(2)$. Verify that $x(t) = -\log(2-t)$ solves this ODE. Does this ODE satisfy local existence, completeness (global existence), uniqueness and continuity with respect to initial conditions? If so, prove it; if not, show why not.

   (b) Let $t^2\ddot{x} - 3t\dot{x} + 4x = 0$ and $x(e) = e^2$. Convert this second order ODE into a first order ODE system. Verify that $x(t) = t^2$ and $x(t) = t^2 \ln t$ both satisfy this ODE. Why isn't this ODE's solution unique?

2. **Two Water Tank Hybrid System**: A commonly cited example in the hybrid system literature is the two water tank system (not to be confused with another common example, the steam boiler). A single input pipe can direct water into either one of two tanks. Each tank supplies water to a separate output pipe. The goal is to switch the input pipe between the two tanks so that the water level in both tanks stays above each tank's minimum level.

   A hybrid automaton description $\mathcal{H}_1$ of the system is

   - Discrete states $Q = \{q_1, q_2\}$, which represent whether the input pipe is currently feeding tank 1 or tank 2.
   - Continuous states $X = \mathbb{R}^2$, where x_1 is the water level in tank 1 and x_2 is the water level in tank 2.
   - Vector field
   $$f(q_1, x) = \begin{bmatrix} w - v_1 \\ -v_2 \end{bmatrix}, \qquad f(q_2, x) = \begin{bmatrix} -v_1 \\ w - v_2 \end{bmatrix}.$$
   - Initialization: start in mode q_1 with water levels x_1^0 and x_2^0.
   - Domain: $\{x \in \mathbb{R}^2 | x_1 \geq r_1 \wedge x_2 \geq r_2\}$, which requires that both tanks remain above their minimum levels.
   - Edges: $e_1 = (q_1, q_2)$ and $e_2 = (q_2, q_1)$, which allow the input pipe to switch between the tanks.
   - Guards: $G(e_1) = (x_2 \leq r_2)$ and $G(e_2) = (x_1 \leq r_1)$, which permit the input pipe to switch if the water level in the tank that is not being replenished drops to its minimum.
   - Reset: the identity (the continuous state is not reset on a mode switch).

   The important features of $\mathcal{H}_1$ are summarized graphically in figure 1. The constants of interest are x_1^0 and x_2^0 (the initial tank levels), v_1 and v_2 (the outflow rates), r_1 and r_2 (the minimum tank levels) and finally w (the inflow rate).

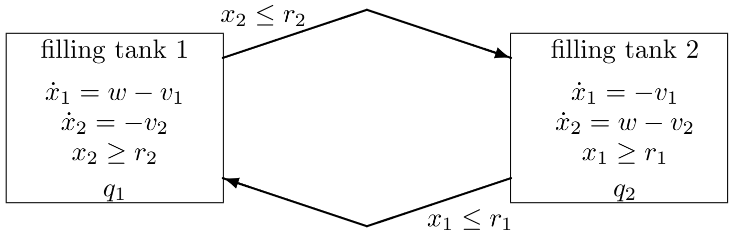

Figure 1: The two water tank hybrid automaton $\mathcal{H}_1$.

(a) Assume that x_1^0, x_2^0, v_1, v_2, r_1, r_2 are fixed. Specify the critical values of w (in terms of these constants) for which the long-term qualitative behavior of the system differs.

(b) Let $x_1^0 = x_2^0 = 5$ liters, $v_1 = v_2 = 2$ liters/second and $r_1 = r_2 = 1$ liter. Write a MATLAB program to simulate the system for $w = \{1, 3, 5\}$ liter(s) / second until $t = 15$ seconds or the domain is violated (either tank drops below its minimum), whichever comes first. For each value of w, provide a time plot of q and x. Use the event detection capabilities of MATLAB's ODE solver suite to handle the discrete switches. If you do not have experience with these features, use the `ballode` example as a starting point.

(c) Discuss how well your simulation results correspond to the true analytic behavior of the hybrid automaton, and how well the automaton models a physical two tank system. Suggest how you might modify the automaton to get more realistic simulation results.

3. **Compute a Reachable Set.** Choose one of the continuous and/or hybrid reachability tools (for example: CheckMate, MATISSE, PHAVer, d/dt, Ellipsoidal Toolbox, Toolbox of Level Set Methods). Choose a model suitable for that tool; it can either be a system we have looked at in class or the readings and/or a (non-trivial) modification of an example system from that tool's documentation.

   (a) Explain your system in a manner suitable for inclusion as an example in a paper about that tool. You will have to describe the system model (graphically and/or mathematically). You will have to relate it to previous work: has it appeared (in the same or modified form) in other publications? Why it is suitable for this tool? You may assume that the reader is familiar with the capabilities of the tool.

   (b) Propose a reachability problem for that system, and approximate the solution of that reachability problem using that tool. Visualize your result, if possible with some sample trajectories.

   (c) Discuss the quality of your result, and how it compares to what you might expect from other tools working on the same problem.

   Write up your results as if they were the example section of a paper on the tool that you are using; the whole writeup should be no more than four pages in a 12 point font (including figures, but you may use an additional page for citations if necessary).

   This is a rather open-ended question, so be careful not to spend too much time on it. In particular, consider your choice of tool carefully before committing—some of the tools that we have discussed in class are no longer maintained and it may be difficult to make them operational.

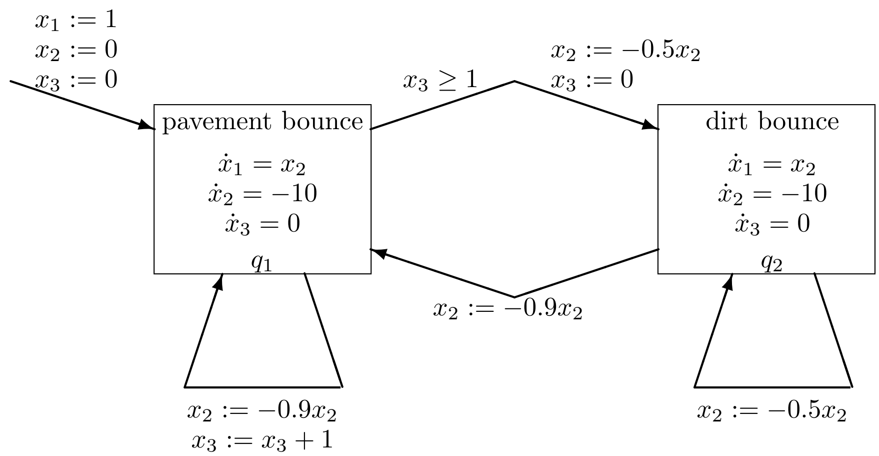

Figure 2: Hybrid Automaton $\mathcal{H}_3$ for a ball bouncing in a parking lot. All modes have domain $x_1 \geq 0$. All mode switches have guards $x_1 \leq 0$ and $x_2 \leq 0$ in addition to those shown.

4. **Another bouncing ball.** Figure 2 shows a hybrid automaton model $\mathcal{H}_3$ of a bouncing ball. In this case the ball is bouncing across an abandoned parking lot, and in many places the pavement is cracked or missing so that dirt or grass is poking through. The two modes of the automaton model whether the last bounce was on pavement (little energy dissipation) or on dirt (significant energy dissipation). Because most of the parking lot is still pavement, the ball will never bounce just once on pavement between dirt bounces; if it bounces on dirt and then on pavement, it will bounce at least once more on pavement before going back to dirt. We model only the vertical motion of the ball in the automaton, so the continuous variables are height x_1, vertical velocity x_2, and a counter for pavement bounces x_3. The initial conditions are shown on the initialization arc.

   (a) Write out the components of the formal model $\mathcal{H}_3 = (Q, X, f, Init, Dom, E, G, R)$.

   (b) Classify $\mathcal{H}_3$ in the four categories that we discussed in class (blocking, deterministic, continuous with respect to initial conditions, zeno).

   (c) Some of the classifications from part (b) make it difficult to simulate $\mathcal{H}_3$ (for example, in Matlab). Briefly describe and justify (with respect to the underlying physical system) how you might resolve those issues in order to create a simulation.