Lagrangian Approaches to Forward Reachability in Continuous State Spaces

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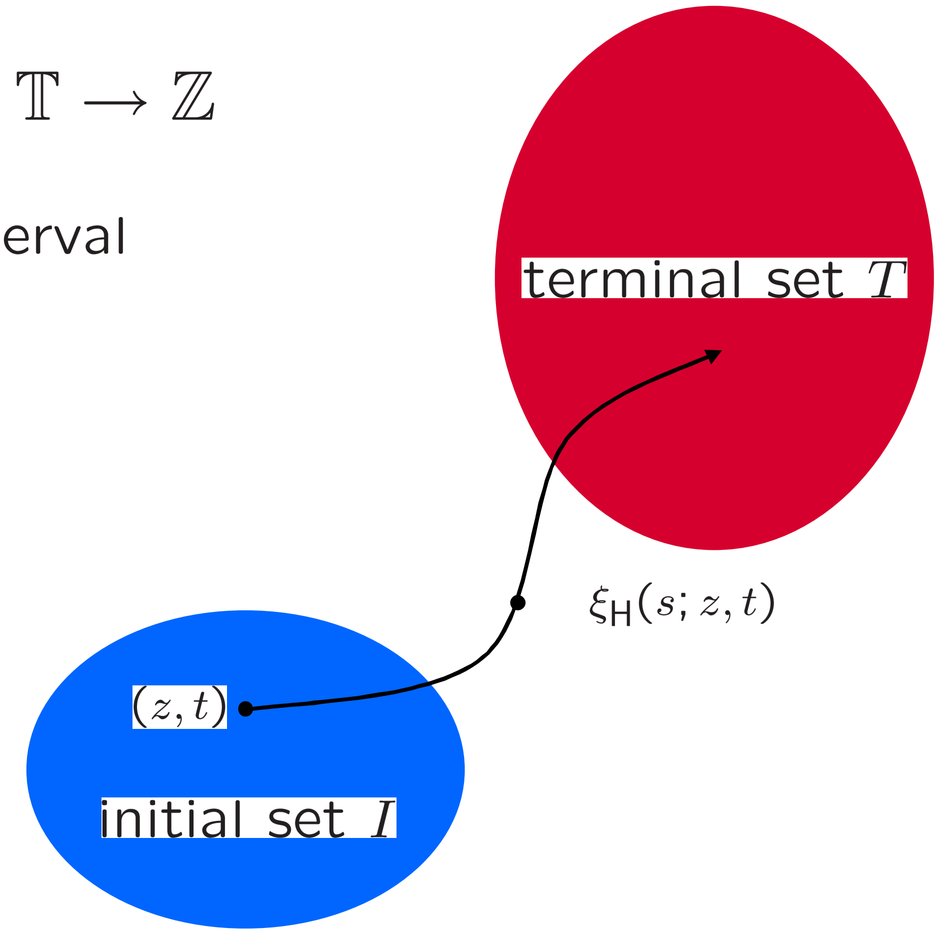
Verification: Safety Analysis

- Does there exist a trajectory of system **H** leading from a state in initial set I to a state in terminal set T?

Trajectory $\xi_{\mathsf{H}}(s; z, t) : \mathbb{T} \rightarrow \mathbb{Z}$

- $\mathbb{T} = [-\mathcal{T}, +\mathcal{T}]$ is time interval
- $\mathbb{Z}$ is state space of H
- $s \in \mathbb{T}$ is current time
- $z \in \mathbb{Z}$ is initial state
- $t \in \mathbb{T}$ is initial time



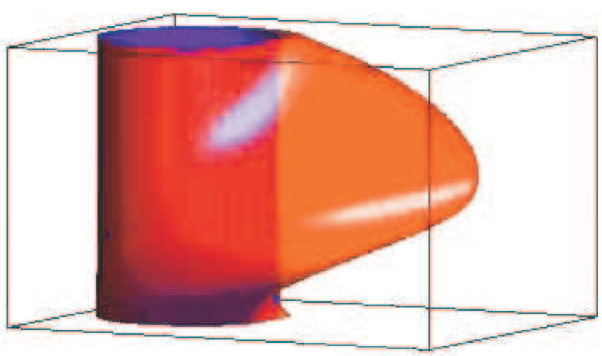
Assumption:

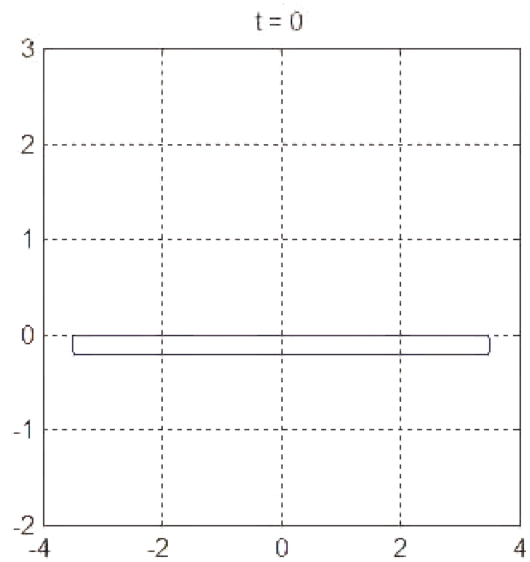
Given z and t

trajectory is unique

Calculating Reach Sets

- Two primary challenges
  - How to represent set of reachable states
  - How to evolve set according to dynamics
- Discrete systems $x_{k+1} = \delta(x_k)$
  - Enumerate trajectories and states
  - Efficient representations: Binary Decision Diagrams
- Continuous systems $dx/dt = f(x)$?





Typical Systems: ODEs

- Common model for continuous state spaces
- Lipschitz continuity of f ensures existence of a unique trajectory
  - Trajectories cannot cross, so boundary of reachable set derives from boundary of initial or target set

ODE $\dot{z}(t) = f(z(t))$
with initial conditions $z(t_0) = z_0$
gives rise to trajectory $\xi_{\mathsf{H}_\mathsf{C}}(t; z_0, t_0)$ where

- $f : \mathbb{Z} \rightarrow \mathsf{T}\mathbb{Z}$ are (Lipschitz) dynamics
- Often $\mathbb{Z} \subseteq \mathbb{R}^{d_z}$

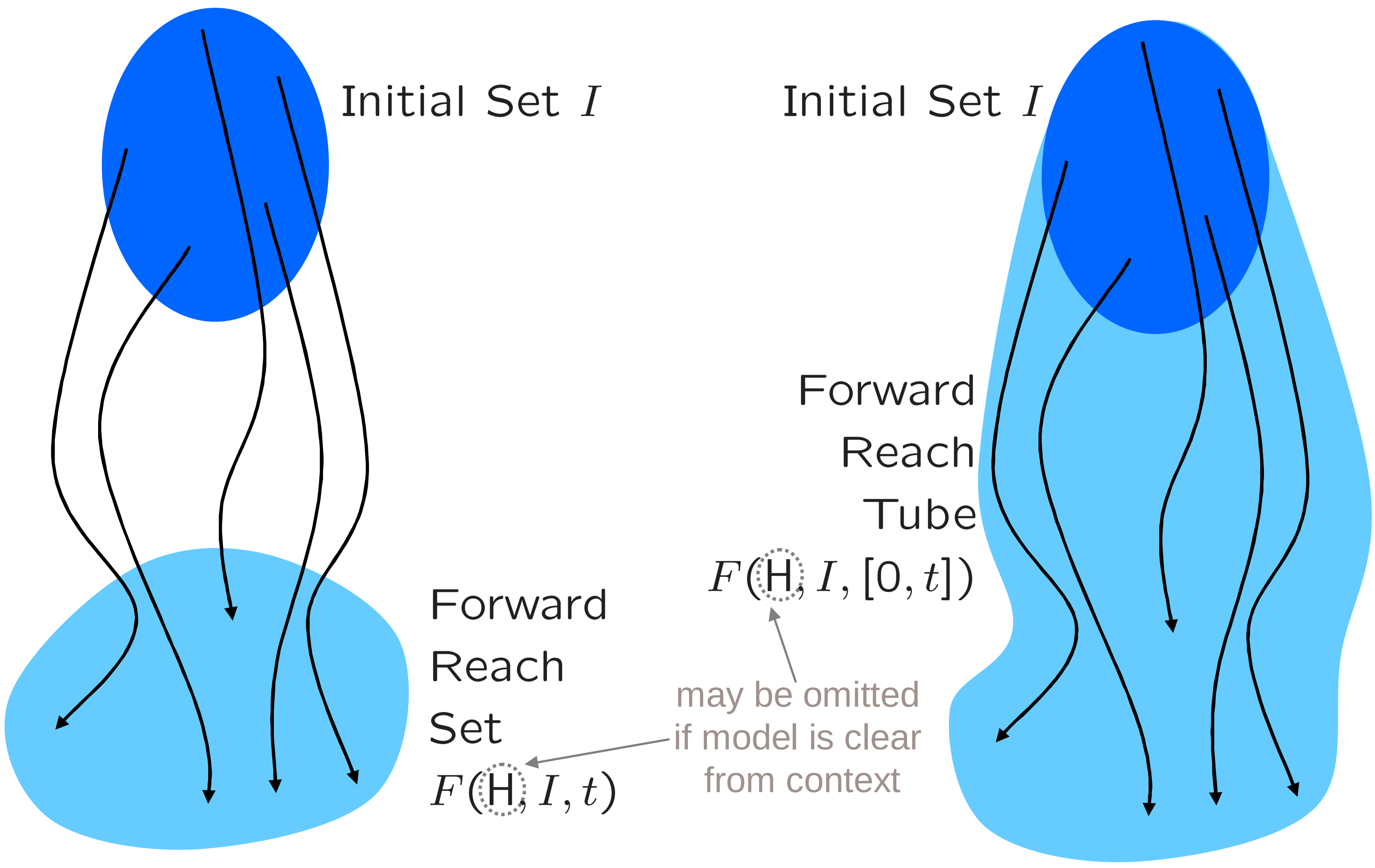
System specified by $\mathsf{H}_\mathsf{C} = (\mathbb{Z}, f)$

Working with Sets

- Optimal control works with a single optimal trajectory
- Verification works with sets of trajectories
  - Takes a nondeterministic (but not probabilistic) viewpoint
- Basic construct is reachability
  - Many versions: forward and backward, sets or tubes
  - When available, what should the input(s) do?
- Many related concepts in control theory
  - Invariant sets, controlled invariant sets, stability
- Safety is not the only verification goal
  - Liveness is a common goal, but often harder to verify

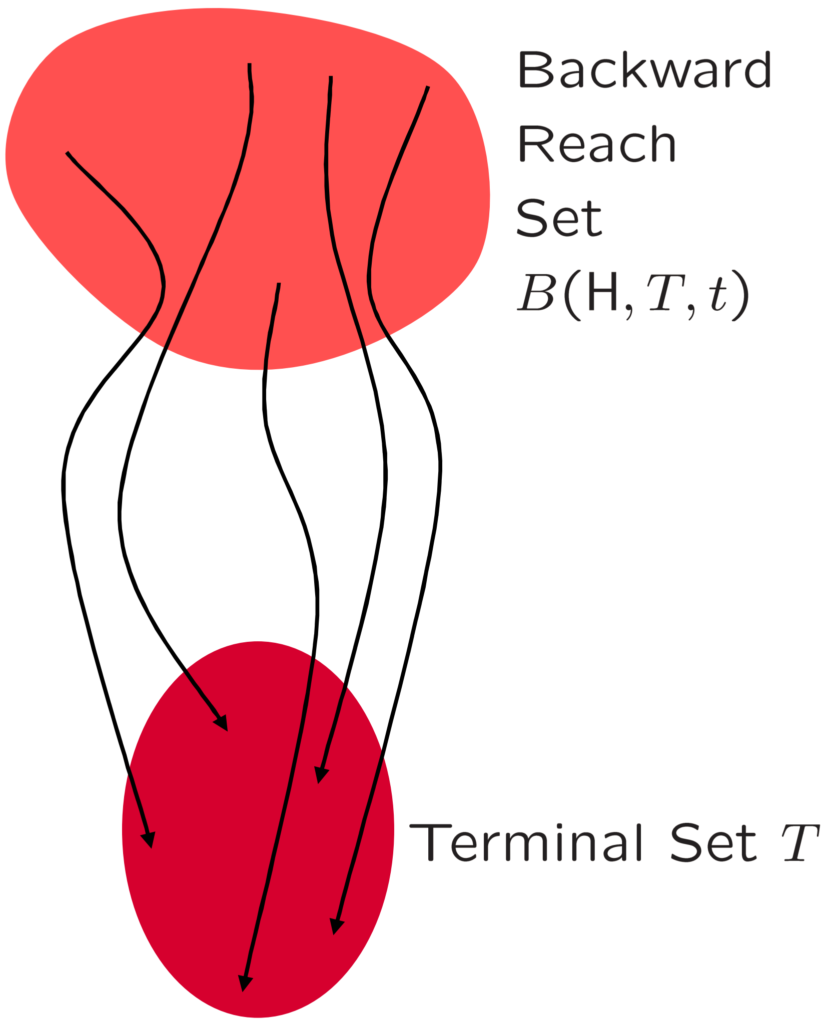
Forward Reachability

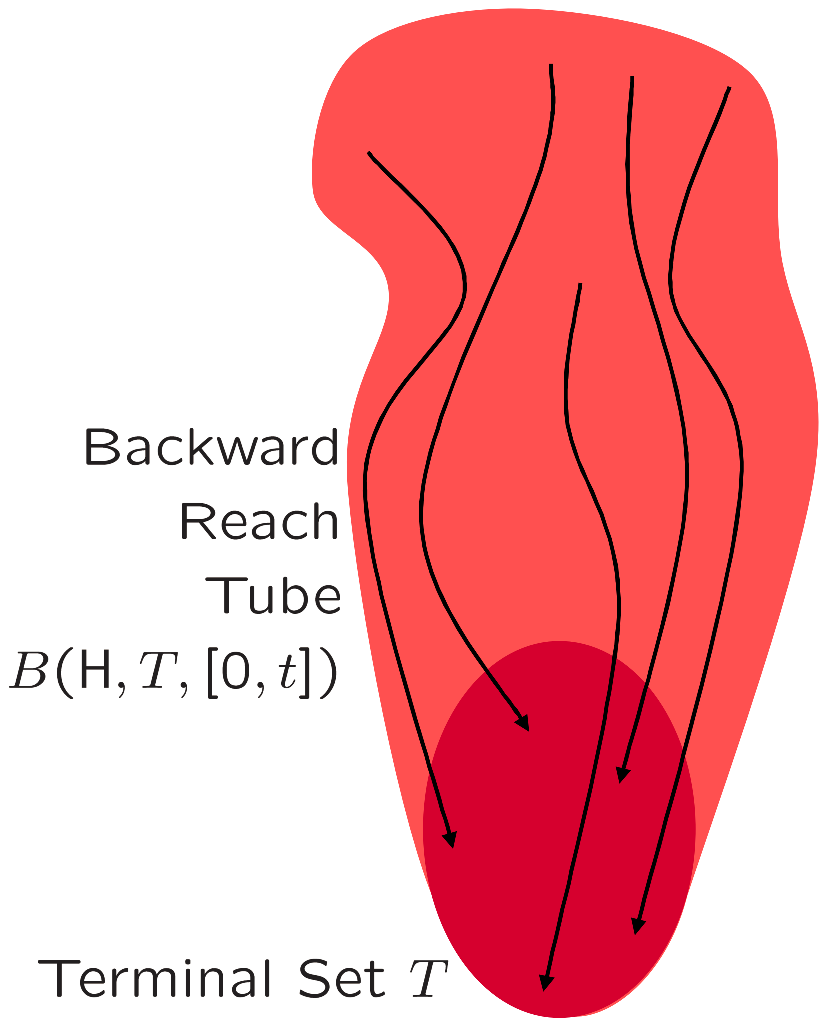
- Start at initial conditions and compute forward



Backward Reachability

- Start at terminal set and compute backwards





Exchanging Algorithms

- Algorithms are (mathematically) interchangeable if system dynamics can be reversed in time

$$\text{Backward dynamic system } \overleftarrow{\mathsf{H}}$$
$$\text{such that } \forall s, t \in \mathbb{T}$$
$$\xi_{\mathsf{H}}(s; z, t) = \hat{z} \Longleftrightarrow \xi_{\overleftarrow{\mathsf{H}}}(s; \hat{z}, t) = z.$$

- For example: $\overleftarrow{\mathsf{H}_{\mathsf{C}}} = (\mathbb{Z}, -f)$
- Then

$$F(\mathsf{H}, S, [0, t]) = B(\overleftarrow{\mathsf{H}}, S, [0, t])$$
$$F(\mathsf{H}, S, t) = B(\overleftarrow{\mathsf{H}}, S, t)$$

Lagrangian Approaches

- "Lagrangian" computation is performed along trajectories of the system
  - Compare with "Eulerian" computation, which occurs on a grid which does not move with the trajectories
- Typically defined in terms of forward reach sets & tubes
- Advantages: Compact representation of sets, overapproximation guarantees, demonstrated high dimensions
- Disadvantages: restricted dynamics, reliance on trajectory optimization, restrictive set representation

Examples of Lagrangian Schemes

- Timed automata
  - Derivatives are zero or one; continuous variables are “stopwatches”
  - Uppaal [Larsen, Pettersson…], Kronos [Yovine,…], …
- Rectangular differential inclusions (“linear” hybrid automata)
  - Derivatives lie in some constant interval
  - Hytech, Hypertech [Henzinger, Ho, Horowitz, Wong-Toi, …]
- Polyhedra and (mostly) linear dynamics
  - Derivatives are linear (or affine) functions
  - Checkmate [Chutinan & Krogh], d/dt [Bournez, Dang, Maler, Pnueli, …], PHAVer [Frehse], Coho [Greenstreet, Mitchell, Yan], others [Bemporad, Morari, Torrisi, …], …
- Ellipsoids and linear dynamics
  - [Botchkarev, Kurzhanski, Kurzhanskiy, Tripakis, Varaiya, …]
- Zonotopes and linear dynamics
  - [Girard, le Guernic & Maler]

Four Examples of Lagrangian Schemes

- CheckMate & convex polygons
- Zonotopes
- Ellipsoids
- Coho & projectagons

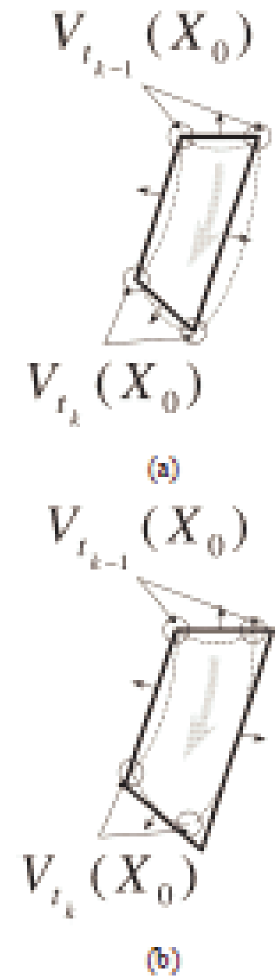
- Note:
  - Choices are heavily influenced by my expertise
  - I may choose different (and potentially conflicting) variable names in these slides when compared with the assigned papers

CheckMate

- Designed to verify properties of Polyhedral Invariant Hybrid Automata (PIHA)
  - Hybrid automata with invariants/guards defined by conjunctions of linear inequalities (convex polyhedra)
- Works by computing an Approximate Quotient Transition System (AQTS)
  - Discrete transition system which conservatively simulates the hybrid automata's evolution
- Released as an add-on to Mathworks' Simulink / Stateflow
  - Model can be constructed graphically
  - Same model can be simulated and verified

Continuous Algorithm

- Start with an initial set X_0
- Reach set $V_{t_k}(X_0)$ at a later time t_k can be determined by simulating trajectories from each vertex of X_0
- Given reach set at t_k and t_{k+1}, initial approximation of reach tube for $[t_k, t_{k+1}]$ is convex hull of $V_{t_k}(X_0)$ and $V_{t_{k+1}}(X_0)$
- Trajectories may curve, so use optimization to push edges of convex hull outward until reach tube contains all reachable states
- For linear dynamics $\dot{x} = \mathbf{A}x$, analytic solution is $\xi(t; x_0, t_0) = e^{\mathbf{A}(t-t_0)}x_0$, so optimization is a linear program for any fixed t (easy to solve)

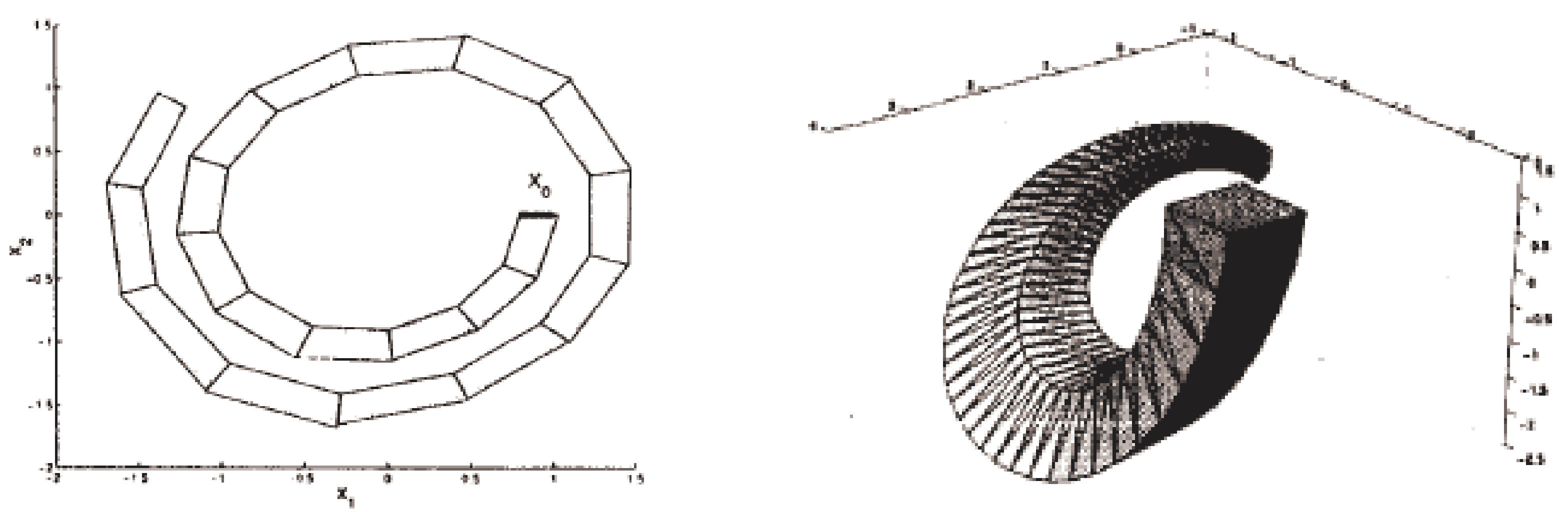


from Chutinan & Krogh,
IEEE Trans. AC,
fig. 4, p. 68 (2003)

Continuous Algorithm's Issues

- Global nonlinear optimization provides no guarantees
  - Dilated convex hull may not contain all possible trajectories
- Trajectories are approximated numerically
- Accomodating inputs requires additional trust in optimization procedure

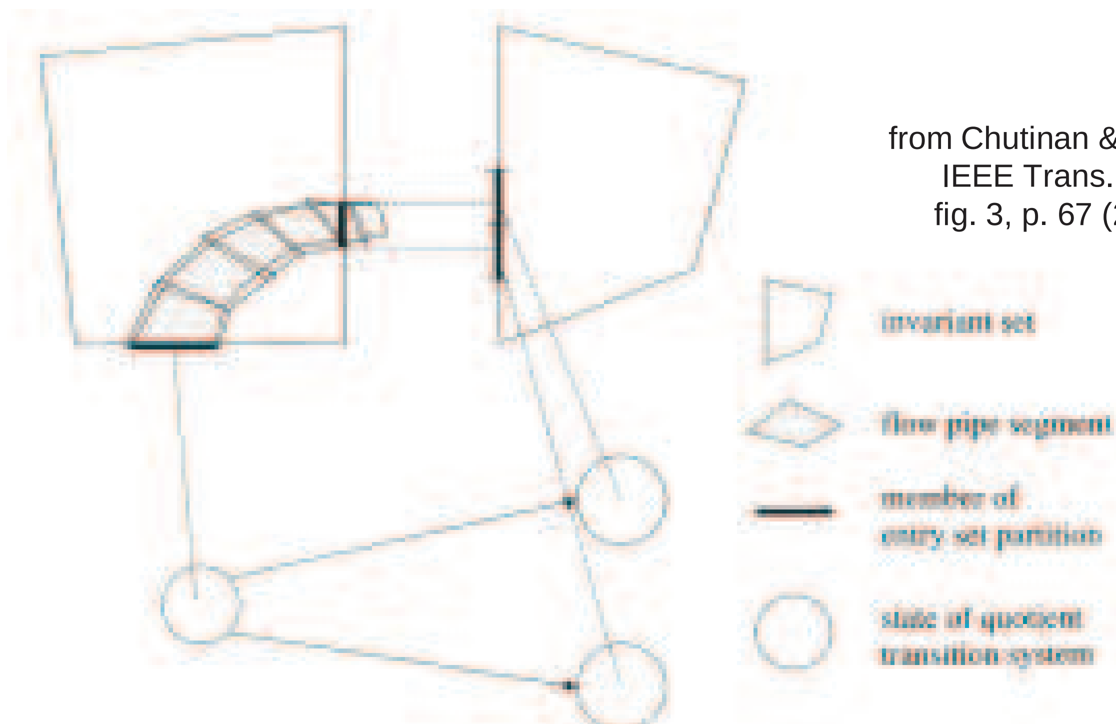
CheckMate reach tube examples for
2D Van der Pol model and a 3D linear model



from Chutinan & Krogh, Proc IEEE CDC, fig. 2, p. 2091 & fig. 3, p 2092 (1998)

Constructing the AQTS

- Reach tube construction is used to determine what set of states on an incoming polyhedral invariant face maps to which set of states on an outgoing polyhedral invariant face
- Sets of states on face are mapped to discrete states in the AQTS (with possible subdivisions)



from Chutinan & Krogh, IEEE Trans. AC, fig. 3, p. 67 (2003)

Primary CheckMate Papers

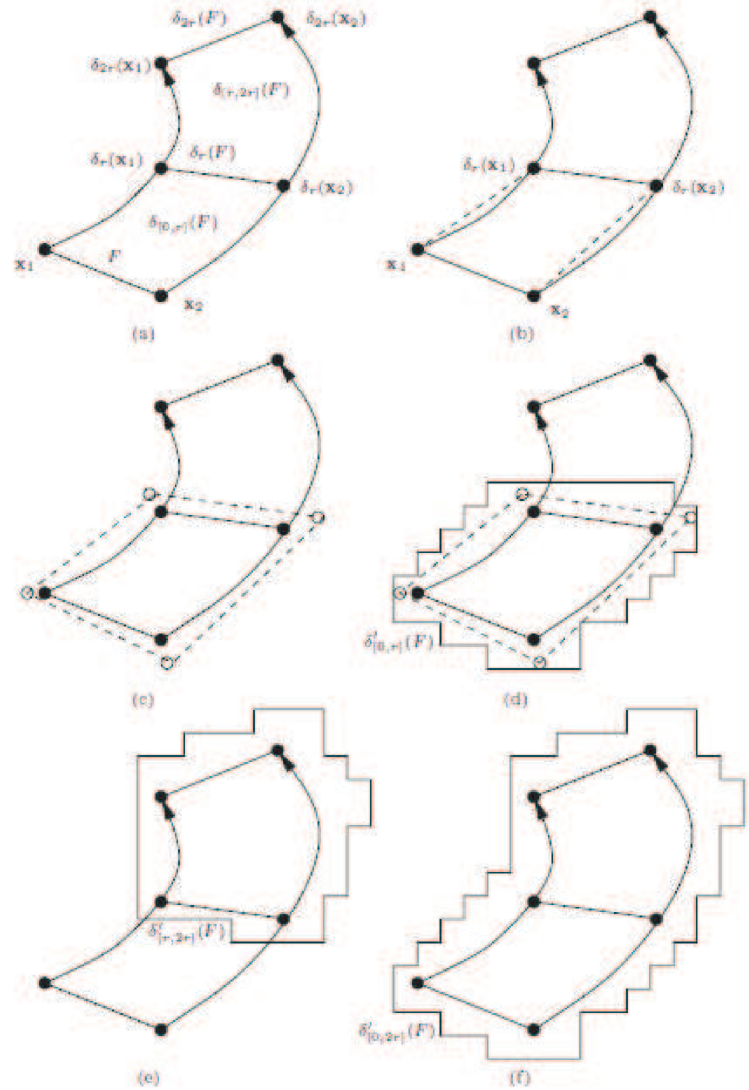
- Alongkrit Chutinan & Bruce H. Krogh, "Computing Polyhedral Approximations to Flow Pipes for Dynamic Systems," *Proc. IEEE Conference on Decision & Control*, pp. 2089–2094 (1998)
  - Details of the scheme for approximating continuous "flow pipes" (forward reach tubes)
- Alongkrit Chutinan & Bruce H. Krogh, "Verification of Infinite-State Dynamic Systems using Approximate Quotient Transition Systems," *IEEE Trans. on Automatic Control*, vol. 46, num. 9, pp. 1401–1410 (2001)
  - Procedure for constructing the AQTS and hence verifying a model for a continuous system, assuming a scheme for computing continuous reachable sets
- Alongkrit Chutinan & Bruce H. Krogh, "Computational Techniques for Hybrid System Verification," *IEEE Trans. on Automatic Control*, vol. 48, num. 1, pp. 64–75 (2003)
  - Journal version of CDC paper, including proof of flow pipe approximation convergence & detailed batch evaporator example
- Numerous other papers (see CheckMate web site)

CheckMate Outcomes

- Most complete tool for hybrid systems with non-constant dynamics
  - (Partially) integrated with commercial design package
  - Handles hybrid system verification, not just continuous reachability
  - Generates counter-examples on failure
  - Later work integrated Counter-Example Guided Abstraction Refinement (CEGAR) [Clarke, Fehnker, Han, Krogh, Stursberg, Theobald, TACAS 2003]
- Unable to move beyond low dimensions
  - Polyhedral representation grows too complex
  - One proposal: Oriented Rectangular Hull representation [Krogh & Stursberg, HSCC 2003]

A Brief Description of d/dt

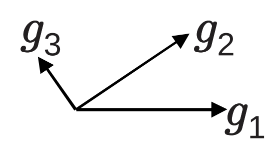
- Similar basic idea to CheckMate
  - Encorporates "griddy polyhedron" construction to control complexity of full reach set representation
  - Various continuous reachability extensions: competing inputs, projections, …
- Many publications
  - Eugene Asarin, Olivier Bournez, Thao Dang & Oded Maler, "Approximate Reachability Analysis of Piecewise-Linear Dynamical Systems" in *Hybrid Systems Computation & Control* (Nancy Lynch & Bruce H. Krogh eds.), LNCS 1790, pp. 20-31 (2000)
  - Fig. 2, p. 25 shown at right



Zonotopes

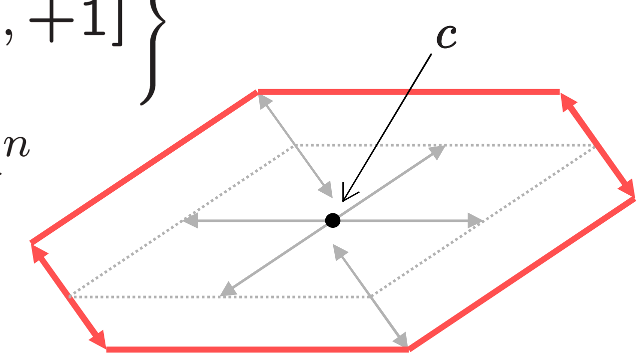
- Representation of general convex polyhedra is too complex in higher dimensional spaces
- Instead, choose a category of sets that can be efficiently represented
- Zonotopes:
  - Image of a hypercube under an affine projection
  - Minkowski sum of a finite set of line segments

$Z = (c, < g_1, \ldots, g_p >)$ denotes



$$Z = \left\{ x \in \mathbb{R}^n \;\middle|\; x = c + \sum_{i=1}^{i=p} \lambda_i g_i,\ \lambda_i \in [-1, +1] \right\}$$

where c, g_1, g_2, ..., g_p are vectors in $\mathbb{R}^n$



$Z = (c, < g_1, g_2, g_3 >)$

Zonotope Features

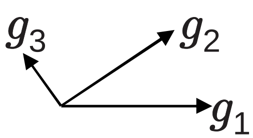
- Compact representation: storage cost $n(p+1)$
- Closed under linear transformation: if $\mathcal{L}x = \mathbf{A}x + b$ then

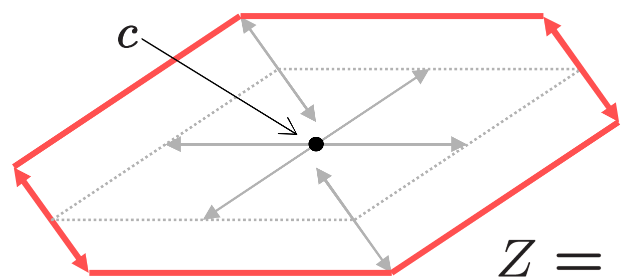
$$\mathcal{L}Z = (\mathcal{L}c, < \mathcal{L}g_1, \ldots, \mathcal{L}g_p >)$$

- Closed under Minkowski sum:

$$Z^{(1)} + Z^{(2)} = (c^{(1)} + c^{(2)}, < g_1^{(1)}, \ldots, g_{p^{(1)}}^{(1)}, g_1^{(2)}, \ldots, g_{p^{(2)}}^{(2)} >)$$

- Conversion to other representations can be expensive; for example, a zonotope may have $2p$ choose $n-1$ facets
- Computation of intersection and union may be difficult; for example, see [Girard & Le Guernic, HSCC 2006]





$$Z = (c, < g_1, g_2, g_3 >)$$

Linear Dynamics with Bounded Inputs

Restrict class of ODEs to the form

$$\dot{x} = \mathbf{A}x + u, \qquad u \in U$$

where U in this case is a hypercube

$$U = \{u \in \mathbb{R}^n \mid \|u\|_\infty \leq \mu\}$$

- $f(x, u) = \mathbf{A}x + u$ is Lipschitz in x, so standard existence and uniqueness results apply
- Trajectories now denoted by $\xi(t; x_0, t_0, u(\cdot))$ where function $u(\cdot) : \mathbb{R} \to U$ is an input signal
- Reach set with fixed (but not necessarily constant) input signal is the same as the input-free case
- Reach set with general input signal is the union over all possible fixed input signals

Continuous Algorithm

- Decompose full reach tube into segments

$$F(I,[0,T]) = \bigcup_i F(I,[ir,(i+1)r])$$

for some small timestep r

- Time-independent ODEs have the semigroup property

$$\xi(t_1 + t_2; x_0, 0) = \xi(t_2; \xi(t_1; x_0, 0), t_1)$$

We can use the semigroup property to deduce

$$F(I,[ir,(i+1)r]) = F(F(I,[(i-1)r, ir]), r)$$

- Therefore, if we can conservatively approximate $F(I,[0,r])$ and $F(Z,r)$ for any Z, we can conservatively approximate $F(I,[0,T])$

Conservative Approximations

- Let $\|\cdot\| = \|\cdot\|_\infty$, "+" for sets be interpreted as the Minkowski sum and $\square(\rho) = \{x \in \mathbb{R}^n \mid \|x\|_\infty \le \rho\}$ (which is a zonotope)
- $F(Z, r) \subseteq e^{r\mathbf{A}} Z + \square(\beta_r)$ where

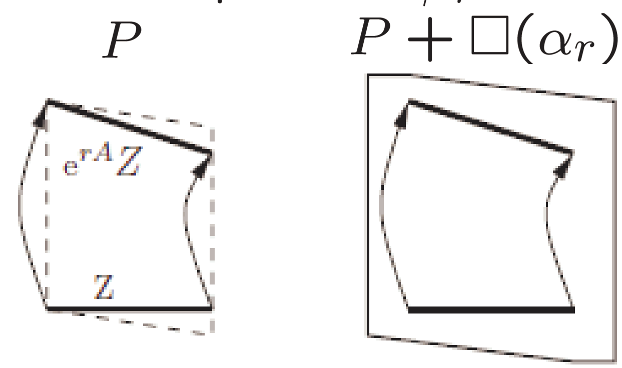
$$\beta_r = \frac{e^{r\|\mathbf{A}\|} - 1}{\|\mathbf{A}\|}\mu,$$

- $F(Z, [0, r]) \subseteq P + \square(\alpha_r + \beta_r)$ where

$$\alpha_r = \left(e^{r\|\mathbf{A}\|} - 1 - r\|\mathbf{A}\|\right) \sup_{x \in Z} \|x\|$$

$$P = \frac{1}{2}\left(c + e^{r\mathbf{A}}c, \left\langle \begin{array}{c} g_1 + e^{r\mathbf{A}}g_1, \dots, g_p + e^{r\mathbf{A}}g_p, \\ c - e^{r\mathbf{A}}c, \\ g_1 - e^{r\mathbf{A}}g_1, \dots, g_p - e^{r\mathbf{A}}g_p \end{array} \right\rangle\right)$$

No inputs $\Rightarrow \beta_r = 0$



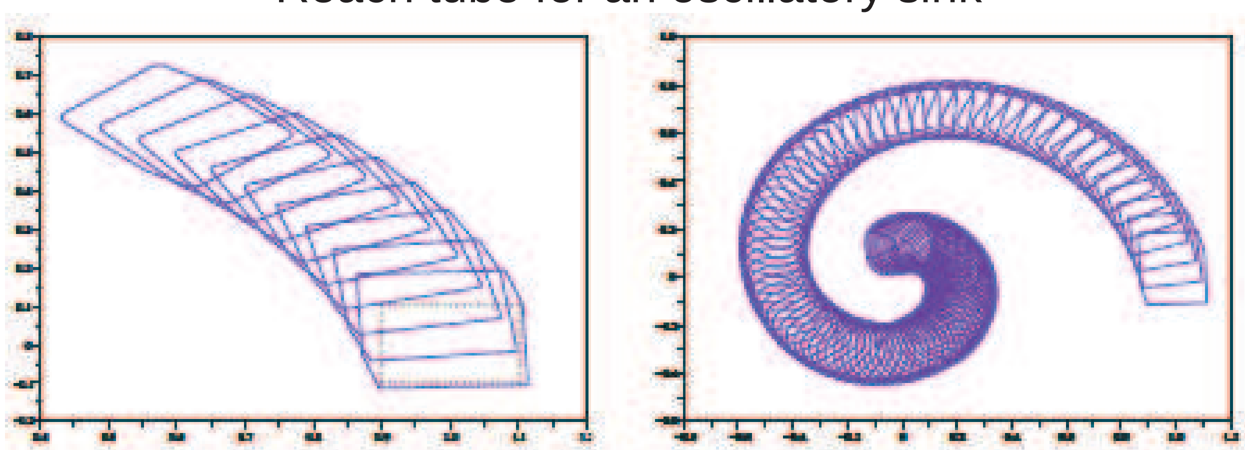
from Girard, HSCC 2005, fig. 2, p. 295

- These approximations can be shown to converge (in the Hausdorff metric) as $r \to 0$

Further Work

- Complexity of zonotopes in basic algorithm grows with time
  - Can conservatively constrain the order of the zonotope
- [Girard, Le Guernic & Maler, HSCC 2006]
  - Refactorizes the Minkowski sum to avoid growth of order
  - Constructs underapproximations and interval hull approximations
  - Discusses extension to hybrid automata (requires set intersection)
- [Girard & Le Guernic, HSCC 2008]
  - "Efficient" Algorithm for zonotope intersection with hyperplane

Reach tube for an oscillatory sink



from Girard, HSCC 2005, fig. 4, p. 298

Zonotope Outcomes

- Still primarily a research project
  - MATISSE tool implements the continuous reachable set computation (including HSCC 2006?)
- Demonstrated on continuous toy examples in dimension 100 (HSCC 2005) and 200 (HSCC 2006)
- Demonstrated on low dimensional hybrid examples
- Zonotope representation has interesting trade-offs
  - Difficulty of computing set intersection and (presumably) union may make abstraction refinement challenging
  - Complexity (zonotope order) can be controlled over a wide range
  - Infinity norm bounds require well scaled system dynamics and inputs

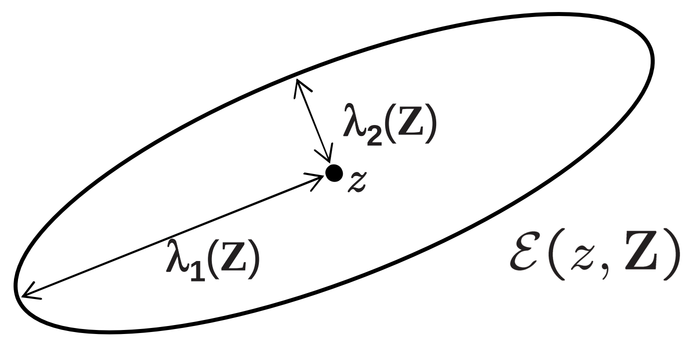
Ellipsoids

- An alternative class of sets which can be efficiently represented in high dimensions
- Represent as the zero level set of a quadratic function
  - So computational costs in a given dimension are similar to LQR or Kalman filtering

Ellipsoid $\mathcal{E}(z, \mathbf{Z}) \subset \mathbb{R}^n$ is specified by

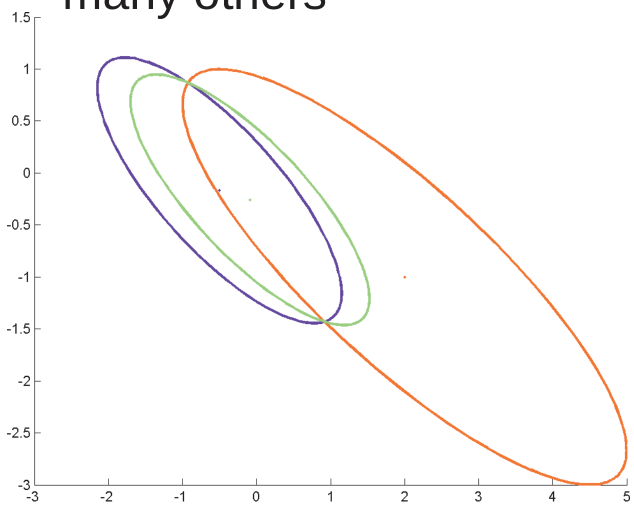
$$\mathcal{E}(z, \mathbf{Z}) = \{x \in \mathbb{R}^d \mid (x - z)^T \mathbf{Z}^{-1}(x - z) \leq 1\}$$

where $\mathbf{Z} \in \mathbb{R}^{n \times n}$ is the symmetric positive definite shape matrix and $z \in \mathbb{R}^n$ is the center

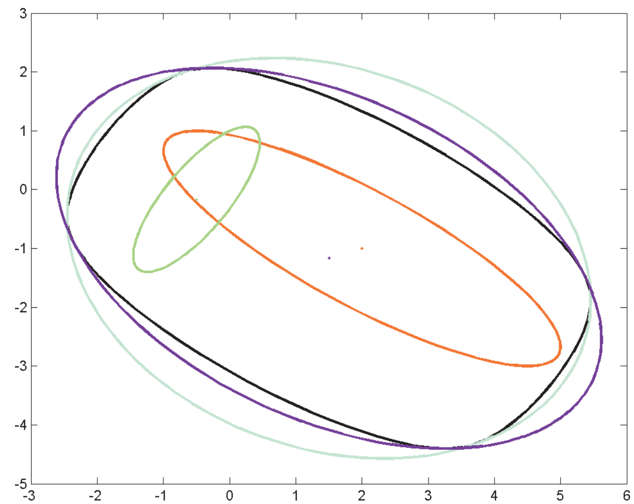


Ellipsoid Features

- Compact representation: ½n^2 + $\mathcal{O}(n)$
- Operations (union, intersection, Minkowski sum, etc.) on ellipsoids rarely give rise to ellipsoids
  - However, inner and/or outer bounding ellipsoids of the results can often be constructed analytically or by convex optimization
  - See various works by Kurzhanski, Kurzhanskiy, Vályi, Varaiya and many others



Two ellipsoids (red & blue) and an ellipsoid bounding their intersection (green)



Two ellipsoids (red & green), their actual Minkowski sum (black), and two ellipsoids bounding their Minkowski sum (cyan & blue)

Ellipsoidal Reachability

- Restrict dynamics to be linear

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$

  where $\mathbf{A}$, $\mathbf{B}$ are matrices, and input $u \in U = \mathcal{E}(p, \mathbf{P})$
- Even if $I = \mathcal{E}(x_0, \mathbf{X}_0)$, reach set $F(I, t)$ is not an ellipse
- It is possible to construct tight external and internal bounding ellipses which touch the reach set at known points $\ell^*(t)$
- Choose $\ell^*(t)$ as a solution to the adjoint of the homogenous dynamics

$$\dot{\ell}^* = -\mathbf{A}^T \ell^* \qquad \text{for some } \ell^*(0) = \ell_0,$$

  so $\ell^*(t) = e^{-\mathbf{A}^T t}\ell_0$
- We can write a recurrance for the tight ellipsoids' parameters

External Bounding Ellipses

- Construct outer bounding ellipsoid $X_\ell^+(t) = \mathcal{E}(x_c(t), \mathbf{X}_\ell^+(t))$ such that $F(I,t) \subseteq X_\ell^+(t)$
- Center is just a trajectory (remember $u \in \mathcal{E}(p, \mathbf{P})$)

$$\dot{x}_c(t) = \mathbf{A}x_c(t) + \mathbf{B}p \qquad x_c(0) = x_0$$

- Shape satisfies a matrix ODE

$$\dot{\mathbf{X}}_\ell^+(t) = \mathbf{A}\mathbf{X}_\ell^+(t) + \mathbf{X}_\ell^+(t)\mathbf{A}^T + \pi_\ell(t)\mathbf{X}_\ell^+(t) + \frac{\mathbf{BPB}^T}{\pi_\ell(t)}$$

$$\mathbf{X}_\ell^+(0) = \mathbf{X}_0$$

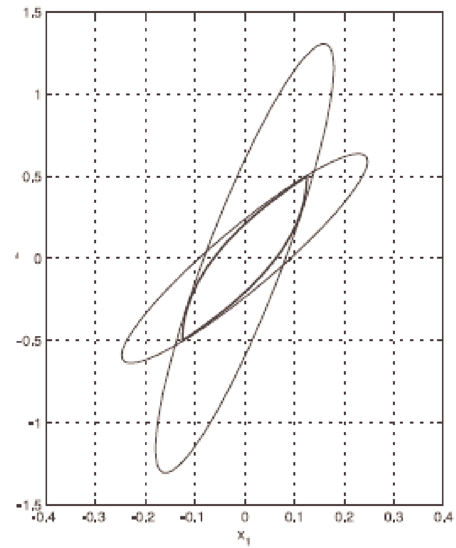
$$\pi_\ell(t) = \left(\frac{\ell^T\mathbf{Y}(t)\mathbf{BPB}^T\mathbf{Y}^T(t)\ell}{\ell^T\mathbf{Y}(t)\mathbf{X}_\ell^+(t)\mathbf{Y}^T(t)\ell}\right)^{\frac{1}{2}}$$

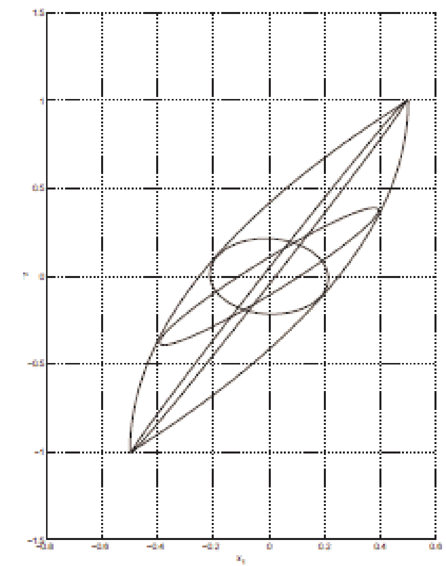
$$\mathbf{Y}(t) = e^{\mathbf{A}t}$$

- A similar recurrance can be defined for an inner ellipsoid $X_\ell^-(t)$ such that $X_\ell^-(t) \subseteq F(I,t)$

Further Work

- Actual derivation allows dynamics and input set to be time-dependent
- Also derived for systems with two inputs: “control” and “disturbance”
  - State x is reachable if there exists an initial condition in I and a feedback control signal $u(\cdot)$ that drives a trajectory to x for every possible disturbance signal $v(\cdot)$
- In practice, compute bounding ellipsoids for several different ℓ
  - For verification, test if all (outer) or any (inner) ellipsoid intersects with the target
  - For visualization and other operations, can compute bounding ellipsoids for intersections and unions
  - Shown at right: two outer and three inner bounding ellipsoids; actual reach set is contained in the intersection of the outer and the union of the inner

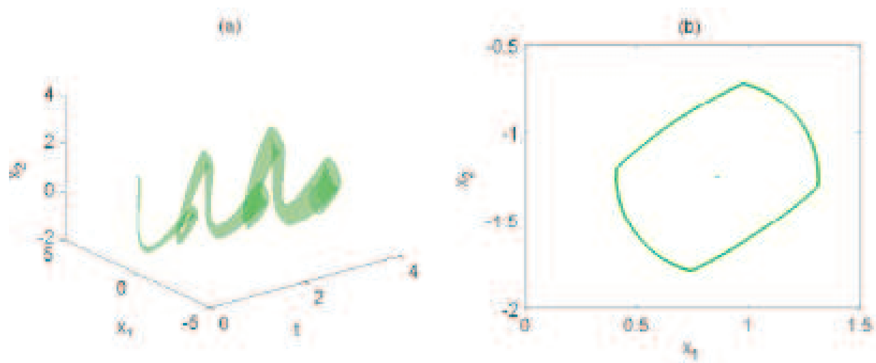




from Kurzhanski & Varaiya, HSCC 2000, fig. 2 & 3, p. 212–213

Ellipsoid Outcomes

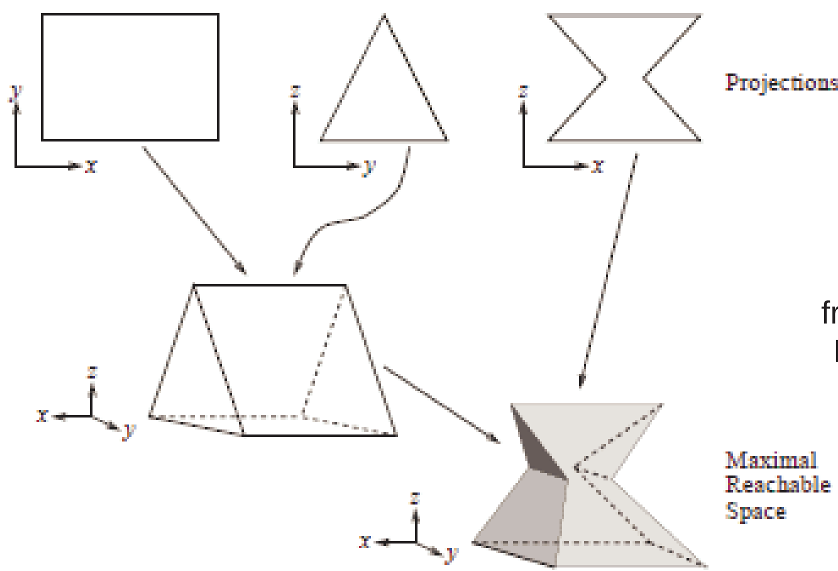
- Described in a whole series of papers by Kurzhanski & Varaiya
- Implemented in Ellipsoidal Toolbox (ET) by Kurzhanskii
  - Documentation for ET provides concise summary of previous work
- Demonstrated in dozens of dimensions
- Demonstrated on low dimensional hybrid problems [Botchkarev & Tripakis, HSCC 2000] and ET
- Ellipsoid representation has different trade-offs
  - Extensive historical work on geometric operations makes extension to hybrid system reachability seem more feasible
  - Complexity of representation cannot be tuned: always $\frac{1}{2}n^2 + \mathcal{O}(n)$
  - General linear input with ellipsoidal bounds adds flexibility



from Kurzhanskii, ET techrep, fig. 7.3, p. 51

Coho & Projectagons

- Two dimensions is easy: Lots of fast, powerful algorithms
  - Can we design an algorithm that primarily works in two dimensional subsets of the full state space?
- “Projectagons”
  - Subset of high dimensional polyhedrons which can be represented as the intersection of a collection of prisms
  - Each prism is the infinite extension (into the other dimensions) of a bounded (potentially nonconvex) two dimensional polygon
  - We actually track only the two dimensional projections



from Greenstreet & Mitchell
HSCC 1999, fig. 1, p. 104

Evolving a Projection (1)

- Let projectagon be P and the prism represented by projection j be P_j, so $P = \cap_j P_j$
- Then CH(P) $\subseteq \cap_j$CH(P_j), where CH(P) is the convex hull of P
- CH(P_j) is easily computed and can be represented by the conjuction of a set of linear inequalities
- Loosen ("bloat") all inequalities by ϵ
- Now consider an individual edge e_i in the two dimensional projection of P_j, which corresponds to a face of P
- Construct a box bounding all states within ϵ of e_i (also a conjuction of linear inequalities)
- The conjunction of all of the inequalities represents all states within ϵ of the face of P corresponding to e_i

Evolving a Projection (2)

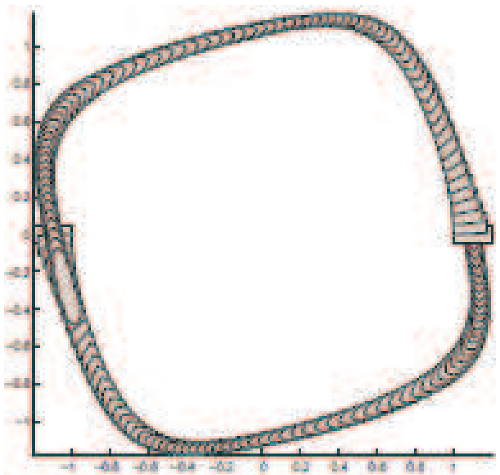
- Construct an affine plus error model $\dot{x} = \mathbf{A}x + b + u$ for $u \in U$ and U a hyperrectangle that is valid within the conjunction of all these linear inequalities
- The forward time mapping of states under this dynamic is linear (if $b = 0$ and $U = \emptyset$ then it is $e^{\mathbf{A}t}$)
- Use linear programming to compute the polygonal projection of the forward time mapping of e_i
- Repeat for all edges in the projection
- Compute the union of all forward time polygons (and all states inside that union)
- Simplify if necessary
- Repeat for all projections
- Repeat for next timestep

Practical Aspects

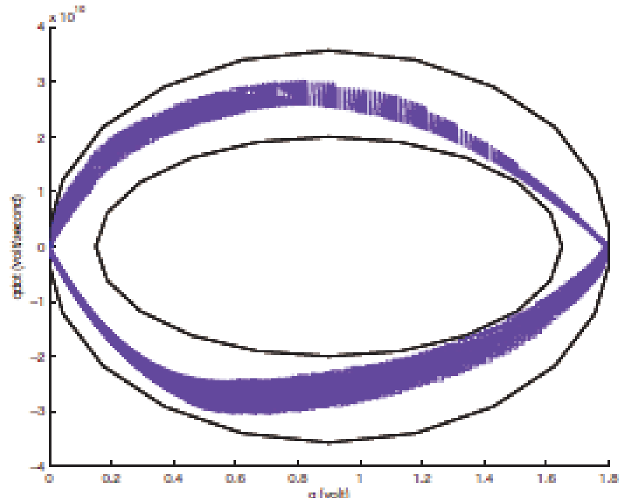
- Geometry and mathematics are well separated
  - Geometry operations in Java, linear programs (LPs) and model computation in Matlab
- LPs are nasty
  - Lots of (nearly) redundant and (nearly) degenerate inequalities
  - Lots of sparsity (only two nonzeros per row)
  - Need to walk the projection (start from nearly optimal point)
  - Need guaranteed optimum for guaranteed overapproximation
  - Led to specialized LP implementation by Laza & Yan: takes advantage of special structure, uses regular floating point calculations to start but guarantees solution accuracy through interval arithmetic and if necessary arbitrary precision arithmetic
- Careful simplification of projections is important
  - Need to keep number of edges under control, but accuracy degrades significantly if nonconvexity is removed
- Choice of projections is not always obvious

Coho Outcomes

- Implemented at UBC in Coho toolset
- Demonstrated on seven dimensional realistic circuit model of a toggle element [Yan & Greenstreet, FMCAD 2007]
  - Included verification of composability to construct a ripple counter
- Projectagons are not as scalable as zonotopes & ellipsoids, but can represent nonconvex reach sets
  - Ample opportunity for parallelization
- Algorithm has considered automatic construction of linear plus error models from nonlinear circuit models



from Greenstreet & Mitchell, HSCC 1999, fig. 6, p. 113



from Yan & Greenstreet, FMCAD 2007, fig. 6, p. 205

Three Other General Approaches

- Eulerian methods (fixed grid reachability)
- State space decomposition (discrete reachability)
- Lyapunov-like methods

Eulerian Approaches

- Time dependent Hamilton-Jacobi
  - Lygeros, Mitchell, Tomlin, Sastry
  - Finite horizon terminal value
  - Continuous implicit representation
- Static Hamilton-Jacobi
  - Falcone, Ferretti, Soravia, Sethian, Vladimirsky
  - Minimum time to reach
  - (Dis)continuous implicit representation
- Viability kernels
  - Saint-Pierre, Aubin, Quincampoix, Lygeros
  - Based on set valued analysis for very general dynamics
  - Discrete implicit representation
  - Overapproximation guarantee
- Backward reachability approach typical of Eulerian algorithms
  - Representation not moving (although it may adapt)
  - Generally handle nonlinear and multiple inputs
  - No examples beyond four dimensions?

State Space Decomposition

- Partition state space and compute reachability over partition
- Examples
  - Uniform grids: Kurshan & MacMillan, Belta and many others
  - Timed Automata “Region Graph”: Alur & Dill
  - Cylindrical Algebraic Decomposition: Tiwari & Khanna
- Advantages: No need to integrate dynamics, direct control over size of representation
- Disadvantages: Restricted classes of dynamics, “wrapping” problem (discrete system has transitions that do not exist in continuous system)

Lyapunov-like Methods

- Invariant sets are isosurfaces of Lyapunov-like functions
- Examples:
  - Convex optimization: Boyd, Hindi, Hassibi
  - Sum of Squares: Prajna, Papachristodoulou, Parrilo
- Advantages: Short certificate proves analytic invariance, no need to integrate dynamics
- Disadvantages: Restricted class of dynamics, no refinement parameter to reduce false negatives, difficult to extract counterexamples

Lagrangian Approaches to Forward Reachability in Continuous State Spaces

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