# Local Consistency in Junction Graphs for Constraint-Based Inference*

**Le Chang** and **Alan K. Mackworth**
Department of Computer Science, University of British Columbia
2366 Main Mall, Vancouver, B.C. Canada V6T 1Z4
{lechang, mack}@cs.ubc.ca

## Introduction

The concept of local consistency plays a central role in constraint satisfaction. Given a constraint satisfaction problem (CSP), local consistency can be characterized as deriving new, possibly tighter, constraints based on local information. The derived constraints simplify the representation of the original CSP without the loss of solutions. This can be seen as a preprocessing procedure. Based on arc consistency (Mackworth. 1977a) for classic CSPs and soft arc consistency (Cooper & Schiex 2004; Bistarelli 2004) for soft CSPs, we presented a weaker condition using a commutative semiring structure to abstract generalized arc consistency (Mackworth 1977b) to handle constraint-based inference (CBI) problems beyond classic and soft CSPs. The weaker condition proposed in (Chang & Mackworth 2005) has also been relaxed to fit generalized approximate preprocessing schemes.

We propose in (Chang & Mackworth 2006) a new family of generalized local consistency concepts for the junction graph representation of CBI problems. Here we provide an extended summary. These concepts are based on a general condition that depends only on the existence and property of the multiplicative absorbing element and does not depend on other semiring properties of CBI problems (Chang & Mackworth 2005). We present several local consistency enforcing algorithms with various levels of enforcement and corresponding theoretic and empirical complexity analyses. Some of these algorithms can be seen as generalized versions of well-known local consistency enforcing techniques in CSPs and can be exported to other domains. Other abstract local consistency concepts are novel to the constraint programming community and provide more efficient preprocessing results. We also discuss the relationship between these local consistency concepts and message passing schemes such as junction tree algorithms and loopy message propagation. Local consistencies can be achieved along with message propagation and improve the efficiency of message passing schemes.

*We thank the reviewers for their comments on this paper. This research was supported by NSERC. Le Chang is a Precarn Scholar and Alan K. Mackworth holds a Canada Research Chair.


## A CBI Framework and Junction Graph

Constraint-Based Inference (CBI) is an umbrella term for a class of various superficially different problems including probabilistic inference, decision-making under uncertainty, CSPs, SATs, decoding problems, and possibility inference. We abstract these problems into a single formal framework (Chang 2005) using an algebraic semiring structure $\mathbf{S} = \langle \mathbf{A}, \oplus, \otimes \rangle$ where constraint combination is represented by the abstract multiplicative operator $\otimes$ and constraint marginalization is represented by the abstract additive operator $\oplus$. A CBI problem $\mathbf{P}$ in this framework is a tuple $(\mathbf{X}, \mathbf{D}, \mathbf{S}, \mathbf{F})$, where $\mathbf{X}$ is a set of variables, $\mathbf{D}$ is a set of finite domains for each variable, $\mathbf{S} = \langle \mathbf{A}, \oplus, \otimes \rangle$ is a commutative semiring, and $\mathbf{F}$ is a set of constraints. Each constraint is a function that maps value assignments of a subset of variables to values in $\mathbf{A}$. Given a CBI problem, the inference task is defined as computing $g_{CBI}(\mathbf{Z}) = \bigoplus_{\mathbf{Y}} \bigotimes_{f \in \mathbf{F}} f$. If $\oplus$ is idempotent, the allocation task is defined as computing $\mathbf{y} = \arg \bigoplus_{\mathbf{Y}} \bigotimes_{f \in \mathbf{F}} f$, where $\arg$ is a prefix of operator $\oplus$. We generalize various exact and approximate inference algorithms (Chang 2005) from different fields based on the CBI framework. Our local consistency concepts proposed in this paper are also based on this CBI framework and apply to CBI problems with commutative semirings that are *eliminative*. Furthermore, if an eliminative semiring is also *monotonic*, these concepts can be modified to fit generalized approximate preprocessing schemes. Details on eliminative and monotonic semirings can be found in (Chang & Mackworth 2005).

A junction graph $\mathcal{J} = (\mathcal{C}, \mathcal{S})$ of a CBI problem $\mathbf{P} = (\mathbf{X}, \mathbf{D}, \mathbf{S}, \mathbf{F})$ is defined as follows: $\mathcal{C} = \{C_1, \cdots, C_n\}$ is a set of clusters, each cluster $C_i$ is an aggregation of variables that is a subset of $\mathbf{X}$ and has attached initially a local constraint $\phi_{C_i} = \mathbf{1}$ ($\mathbf{1}$ is the multiplicative identity element s.t. $\mathbf{1} \otimes a = a, \forall a \in \mathbf{A}$); $\mathcal{S} = \{S_{ij} | C_i, C_j \in \mathcal{C}\}$ is a set of separators between $C_i$ and $C_j$ if $C_i \cap C_j \neq \emptyset$ and $S_{ij}$ is an aggregation of variables that consists of the intersection of $C_i$ and $C_j$. A junction graph satisfies the condition that for any constraint $f \in \mathbf{F}$, there exists a cluster $C_i \in \mathcal{C}$ s.t. $Scope(f) \subseteq C_i$. The definition of junction graph ensures that the subgraph induced by any variable is connected. We say a junction graph is *initialized* if for each constraint $f \in \mathbf{F}$, we choose a cluster $C_i$ s.t. $Scope(f) \subseteq C_i$ and update $\phi_{C_i}$ by $\phi_{C_i} \otimes f$.

## Local Consistency for CBI Problems

We present here novel local consistency concepts for initialized junction graphs of a CBI problem with an eliminative semiring. If a semiring is both eliminative and monotonic, it is straightforward to modify these concepts as approximate local consistencies using an element $\epsilon \in \mathbf{A}$ to approximate the multiplicative absorbing element $\alpha_\otimes$ that is equal to the additive identity element $\mathbf{0}$ for an eliminative semiring, and using $\leq_\mathbf{S}$ to replace $\neq$ in the following definitions. The fundamental concept of local consistency for an initialized junction graph of a CBI problem is *single cluster consistency*.

**Definition 1 (Single Cluster Consistency (SCC))** *A cluster $C_i$ of an initialized junction graph is locally consistent if $\forall X \in Scope(\phi_{C_i})$, $\forall x \in \mathbf{D}_X$, $\exists \mathbf{w}$, a value assignment of variables $Scope(\phi_{C_i})_{-X}$, s.t. $\phi_{C_i}(x, \mathbf{w}) \neq \alpha_\otimes$. An initialized junction graph of a CBI problem is SCC if all the clusters are locally consistent.*

Single cluster consistency covers the definition of Generalization of Generalized Arc Consistency (GGAC) (Chang & Mackworth 2005), which abstracts generalized arc consistency in constraint programming. If the junction graph is primal, SCC is identical to GGAC. If the junction graph is constructed without satisfying this special structural requirement, SCC is stronger than GGAC in general. We also introduce two other stronger local consistencies: Directional and Neighborhood Cluster Consistencies. Details of the corresponding exact and approximate cluster consistency enforcing algorithms can be found in (Chang & Mackworth 2006).

**Definition 2 (Directional Cluster Consistency (DCC))** *Given a total ordering $\mathcal{OC}$ of the clusters and a cluster $C_i$, let $S_{ij}$ be a separator between cluster $C_j$ and $C_i$ and $\mathbf{L}(C_i)$ be a subset of clusters that consist of lower order neighbor clusters of $C_i$. Let $g_i = \phi_{C_i} \otimes \bigotimes_{C_j \in \mathbf{L}(C_i)} (\bigoplus_{C_j - S_{ij}} \phi_{C_j})$. We say $C_i$ is directional consistent if $\forall X \in Scope(g_i)$, $\forall x \in \mathbf{D}_X$, $\exists \mathbf{w}$, a value assignment of variables $Scope(g_i)_{-X}$, s.t. $g_i(x, \mathbf{w}) \neq \alpha_\otimes$. An initialized junction graph of a CBI problem is directional cluster consistent w.r.t. $\mathcal{OC}$ if all clusters are directional consistent.*

**Definition 3 (Neighborhood Cluster Consistency (NCC))** *Given a cluster $C_i$ of an initialized junction graph, Let $\mathbf{N}(C_i)$ be a subset of clusters that are neighbor clusters of $C_i$. Let $g_i = \phi_{C_i} \otimes \bigotimes_{C_j \in \mathbf{N}(C_i)} (\bigoplus_{C_j - S_{ij}} \phi_{C_j})$. We say $C_i$ is neighborhood consistent if $\forall X \in Scope(g_i)$, $\forall x \in \mathbf{D}_X$, $\exists \mathbf{w}$, a value assignment of variables $Scope(g_i)_{-X}$, s.t. $g_i(x, \mathbf{w}) \neq \alpha_\otimes$. An initialized junction graph of a CBI problem is neighborhood cluster consistent if all clusters are neighborhood consistent.*

## Complexities and Discussion

The worst case space complexities of all three local consistency enforcing algorithms are the same: linear in the number of clusters in the junction graph and exponential in the maximal cluster size. The worst case time complexities are also linear in the size of the junction graph and exponential in maximal cluster size. We compare their upper bounds for time and space in Table 1. All of them use the same space, though achieving SCC uses the least time, followed by DCC, and then NCC. We show the experimental results of applying the approximate variants of these algorithms to both Weighted CSPs and Probability Assessment problems in (Chang & Mackworth 2006).

| | SCC | DCC | NCC |
|---|---|---|---|
| Time | $\lvert\mathcal{C}\rvert d^{k+1}$ | $(\lvert\mathcal{S}\rvert + \lvert\mathcal{C}\rvert)d^{k+1}$ | $(2\lvert\mathcal{S}\rvert + \lvert\mathcal{C}\rvert)d^{k+1}$ |
| Space | $\lvert\mathcal{C}\rvert d^{k+1}$ | $\lvert\mathcal{C}\rvert d^{k+1}$ | $\lvert\mathcal{C}\rvert d^{k+1}$ |

Table 1: Time and space upper bound comparison among various local consistency enforcing algorithms for a junction graph $\mathcal{J} = (\mathcal{C}, \mathcal{S})$ of a given CBI problem, where $d = \max_{D_i \in \mathbf{D}} |D_i|$ and $k = \max_{C_i \in \mathcal{C}} |C_i|$.

Given the identical message representation and updating scheme in the junction tree (JT) algorithm (Shenoy & Shafer 1990) and DCC enforcing, it is straightforward to show that DCC can be achieved along with the inward message passing in the JT algorithm. Loopy message propagation (LMP) (Murphy, Weiss, & Jordan 1999) is another widely studied approximate inference approach based on the junction graph representation in probability inferences. NCC can be achieved along with each message updating step in the LMP without additional computational cost except invalid value detection at each cluster. The time and space complexities of both JT and LPM are reduced after the preprocessing following DCC and NCC enforcement, respectively. Detailed discussion can be found in (Chang & Mackworth 2006).

## References


Bistarelli, S. 2004. *Semirings for Soft Constraint Solving and Programming*. Springer-Verlag.

Chang, L., and Mackworth, A. K. 2005. A generalization of generalized arc consistency: From constraint satisfaction to constraint-based inference. In *IJCAI05 Workshop on Modelling and Solving Problems with Constraints*, 68–75.

Chang, L., and Mackworth, A. K. 2006. Local consistency in junction graphs for constraint-based inference. Technical Report TR-2006-08, Dept. of Computer Science, Univ. of British Columbia. http://www.cs.ubc.ca/cgi-bin/tr/2006/TR-2006-08.

Chang, L. 2005. Generalized constraint-based inference. Master's thesis, Dept. of Computer Science, Univ. of British Columbia.

Cooper, M., and Schiex, T. 2004. Arc consistency for soft constraints. *Artificial Intelligence* 154(1-2):199–227.

Mackworth., A. K. 1977a. Consistency in networks of relations. *Artificial Intelligence* 8:99–118.

Mackworth, A. K. 1977b. On reading sketch maps. In *IJCAI77*, 598–606.

Murphy, K. P.; Weiss, Y.; and Jordan, M. I. 1999. Loopy belief propagation for approximate inference: An empirical study. In *UAI99*, 467–475.

Shenoy, P. P., and Shafer, G. 1990. Axioms for probability and belief-function propagation. In *UAI90*. 169–198.