Multiple Path Pruning, Iterative Deepening and IDA*

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UBC CS 322 - Search 7

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Textbook § 3.7.1-3.7.3

Lecture Overview

- Some clarifications & multiple path pruning
- Recap and more detail: Iterative Deepening and IDA*

Clarifications for the A* proof

- Defined two lemmas about prefixes x of a solution path s
  - (I called the prefix pr, but a 2-letter name is confusing; let's call it x instead)

- Clarifications:
  - "Lemma":
    proven statement, stepping stone in larger proof
  - "Prefix" x of a path s:
    subpath starting from the same node as s
    - E.g. s=(a,c,z,e,d), short aczed
    - All prefixes x: a, ac, acz, acze, aczed
    - E.g. not a prefix: ab, ace, acezd (order is important!)

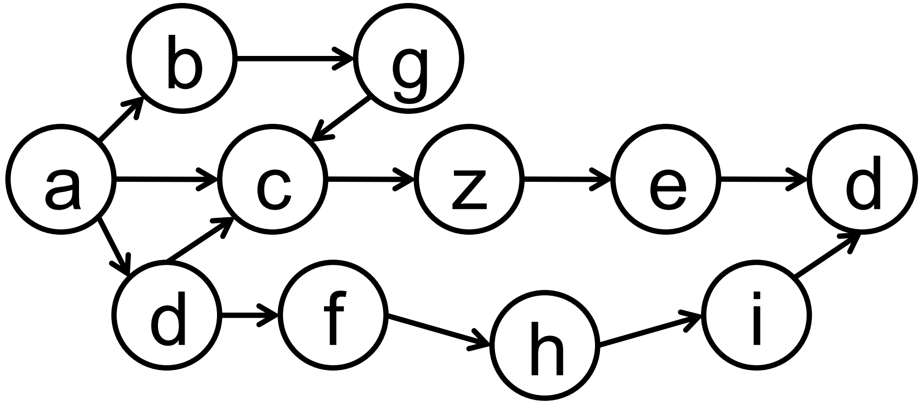
Prefixes

- Which of the following are prefixes of the path aiiscool?

aicool | ai | aii | aisc

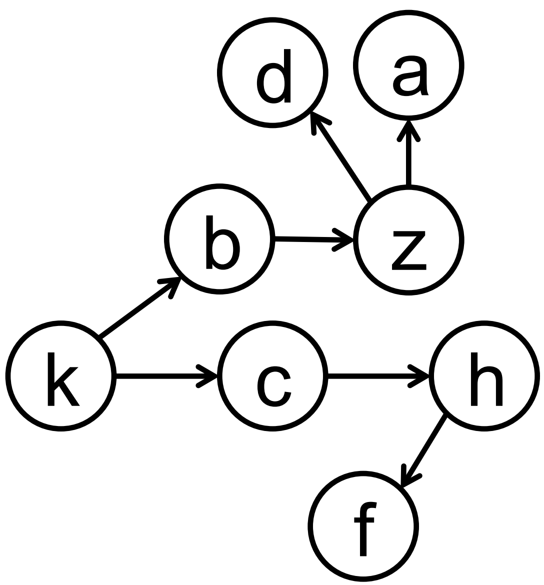
- ai and aii
- aiisc is different from aisc !
  - The optimal solution won't have a cycle if all path costs are > 0

Recap: A* admissibility

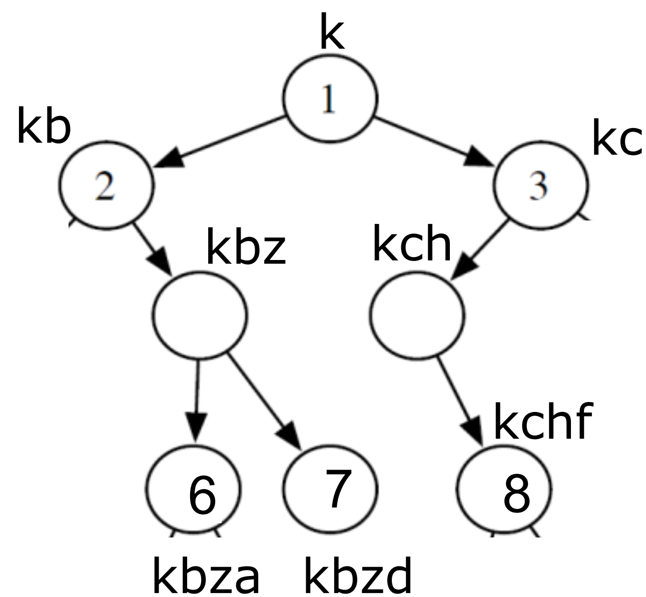


- f_{min}:= cost of an optimal solution path s (e.g. s=aczed)
  - Cost is unknown but finite if a solution exists
- Lemmas for prefix x of s (exercise: prove at home)
  - Has cost $f(x) \leq f_{min}$ (due to admissibility)
  - Always one such x on the frontier (by induction)
- Used these Lemmas to prove:
  A* only expands paths x with $f(x) \leq f_{min}$
- Then we're basically done!
  - Only finite number of such paths ($\Rightarrow$ completeness)
  - Solution with cost > f_{min} won't be expanded ($\Rightarrow$ optimality)

Clarification: state space graph vs search tree



State space graph.

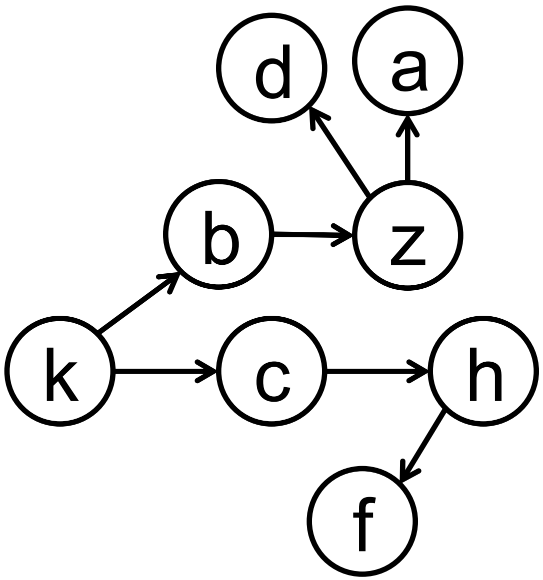


Search tree.

Nodes in this tree correspond to paths in the state space graph

If there are no cycles, the two look the same

Clarification: state space graph vs search tree



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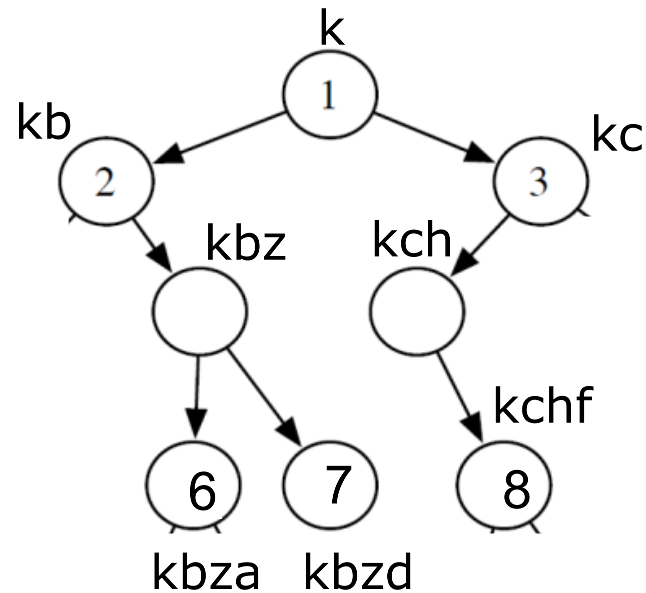


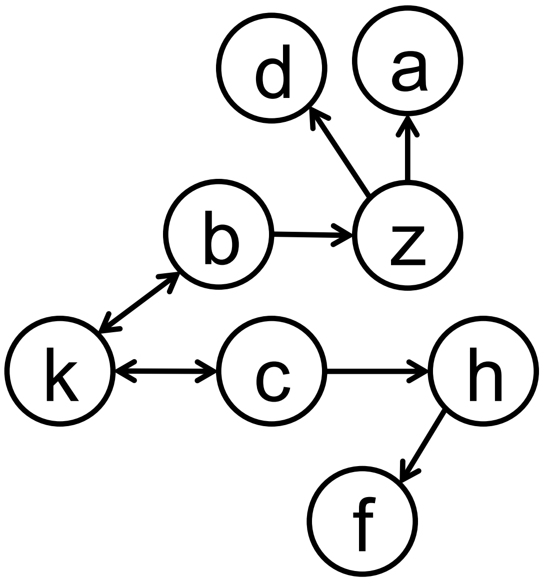
Search tree.

What do I mean by the numbers in the search tree's nodes?

Node's name

Order in which a search algo. (here: BFS) expands nodes

Clarification: state space graph vs search tree



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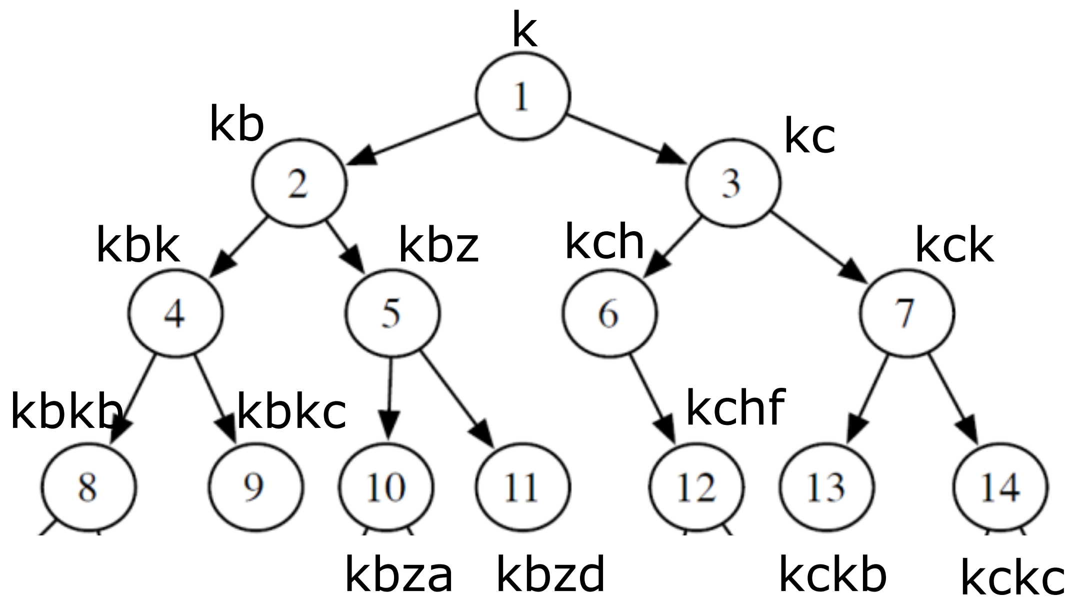
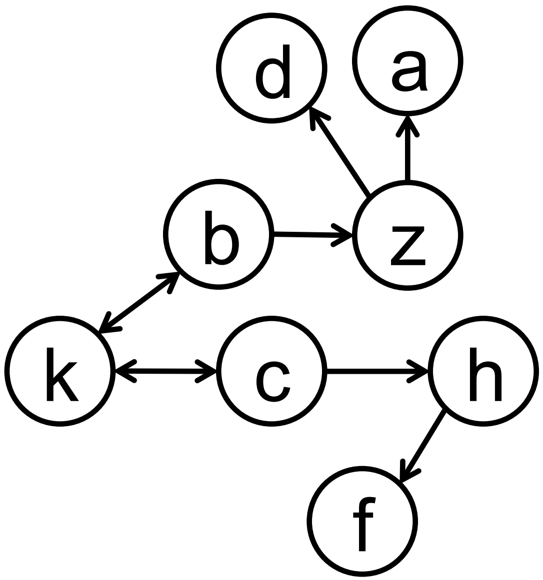


Search tree.
(only first 3 levels, of BFS)

- If there are cycles, the two look very different

Clarification: state space graph vs search tree



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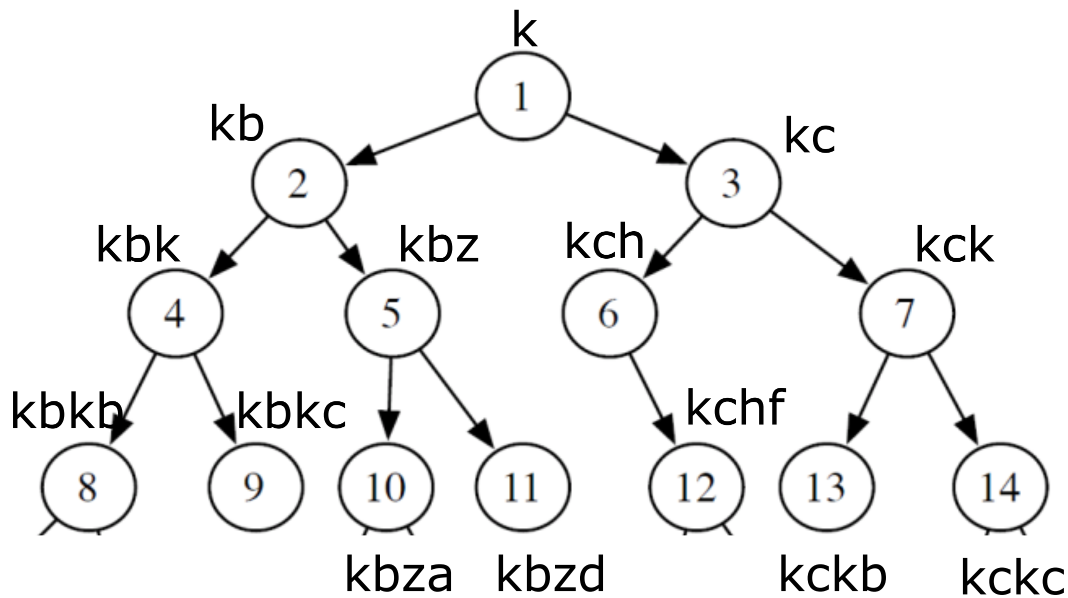
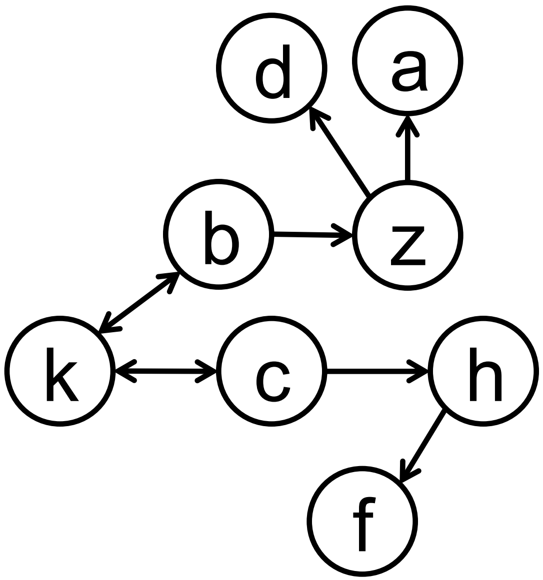


Search tree.
(only first 3 levels, of BFS)

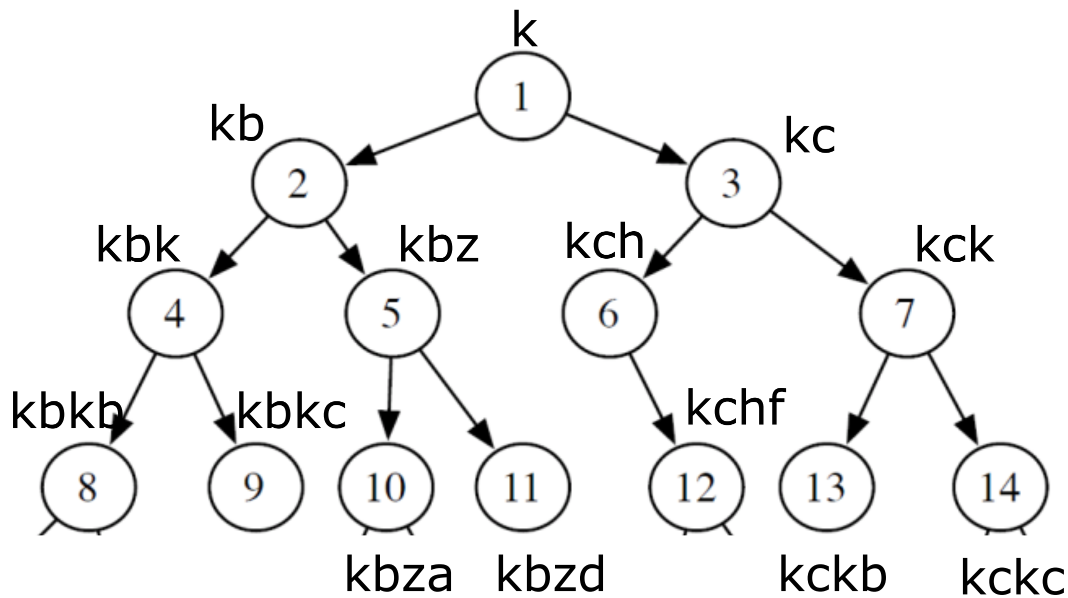
What do nodes in the search tree represent in the state space?

nodes | edges | paths | states

Clarification: state space graph vs search tree



State space graph.



Search tree.
(only first 3 levels, of BFS)

What do edges in the search tree represent in the state space?

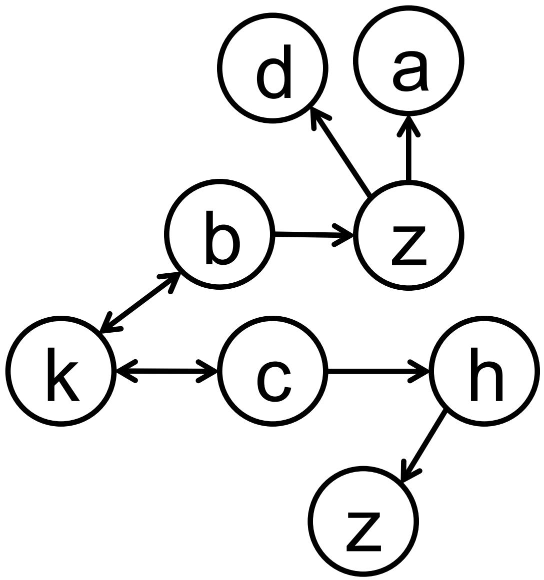
nodes

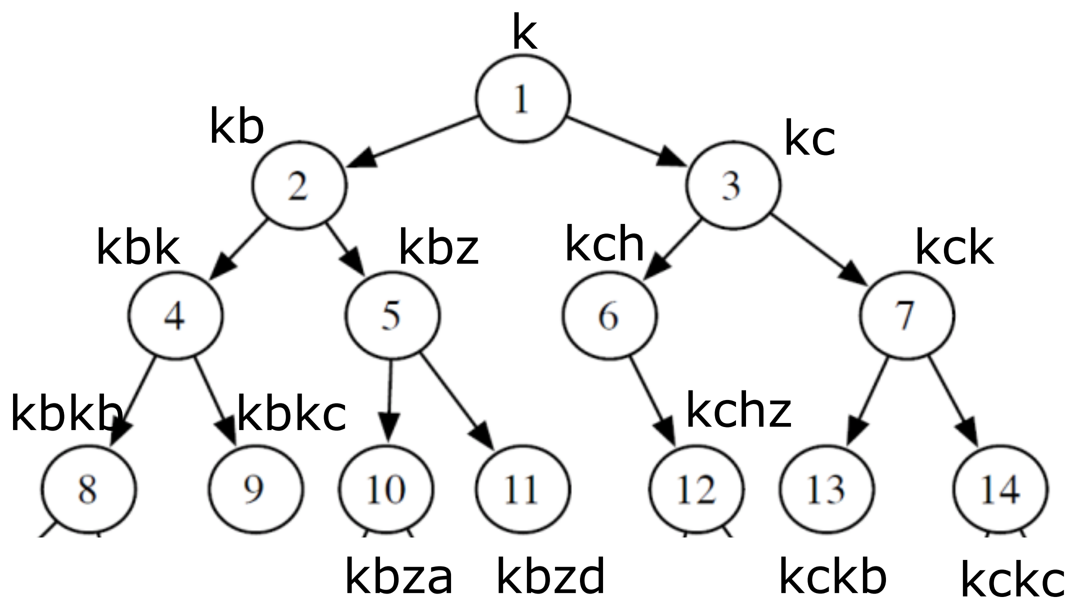
edges

paths

states

Clarification: state space graph vs search tree





State space graph.

Search tree.

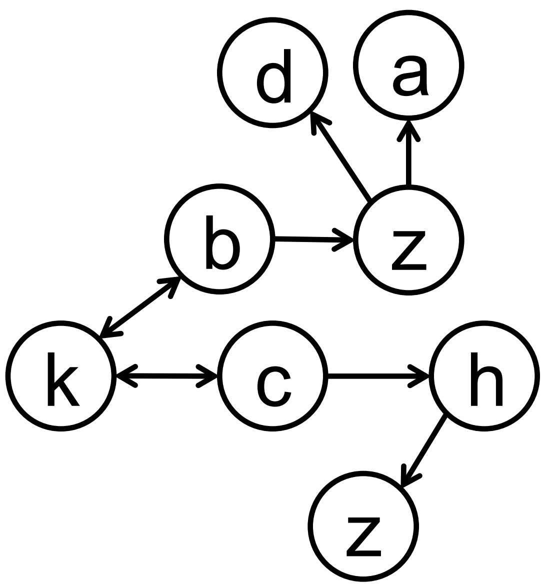
Nodes in this tree correspond to paths in the state space graph

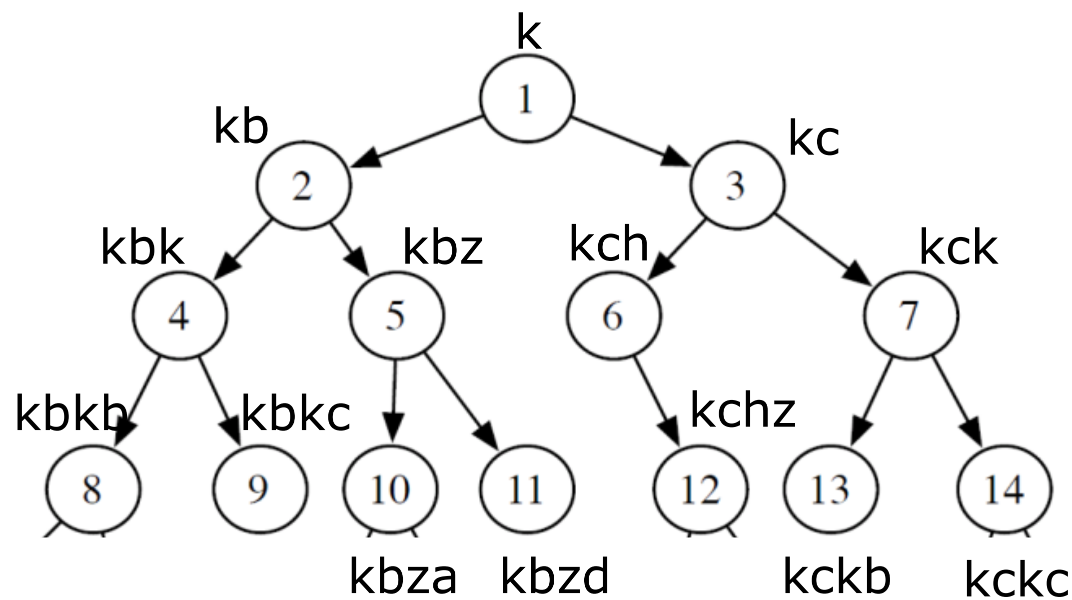
(if multiple start nodes: forest)

May contain cycles!

Cannot contain cycles!

Clarification: state space graph vs search tree





State space graph.

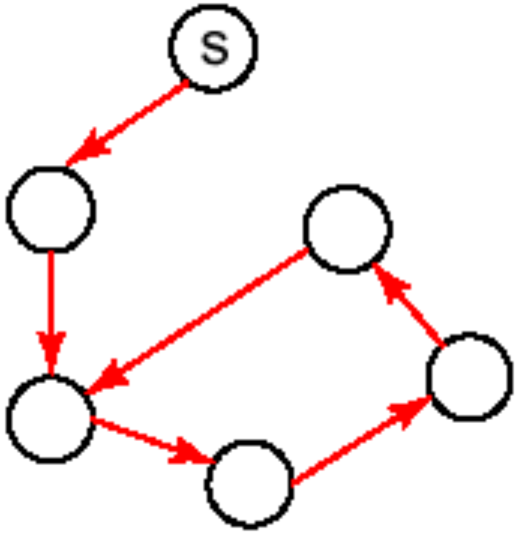
Search tree.

Nodes in this tree correspond to paths in the state space graph

Why don't we just eliminate cycles?

Sometimes (but not always) we want multiple solution paths

Cycle Checking: if we only want optimal solutions



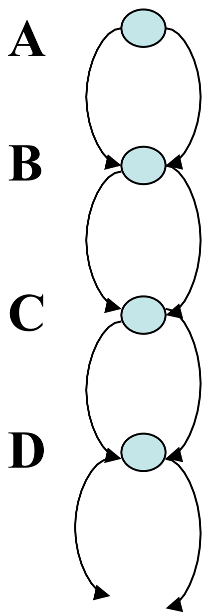
- You can prune a node n that is on the path from the start node to n.
- This pruning cannot remove an optimal solution ⇒ cycle check

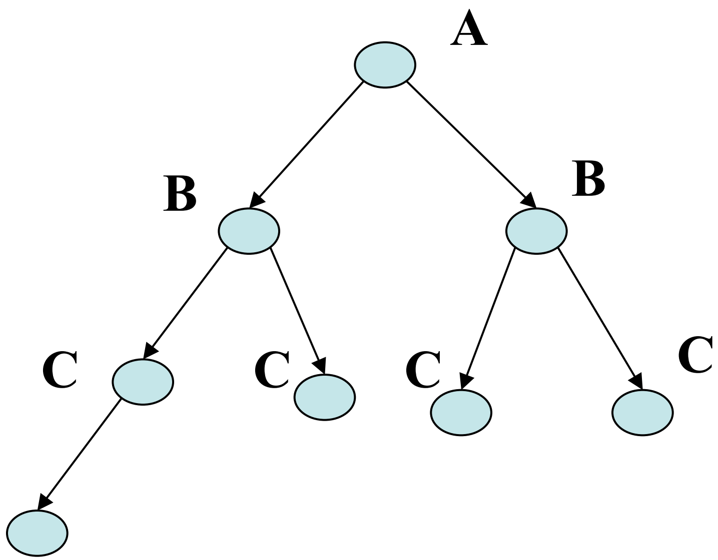
- Using depth-first methods, with the graph explicitly stored, this can be done in constant time
  - Only one path being explored at a time

- Other methods: cost is linear in path length
  - (check each node in the path)

Size of search space vs search tree

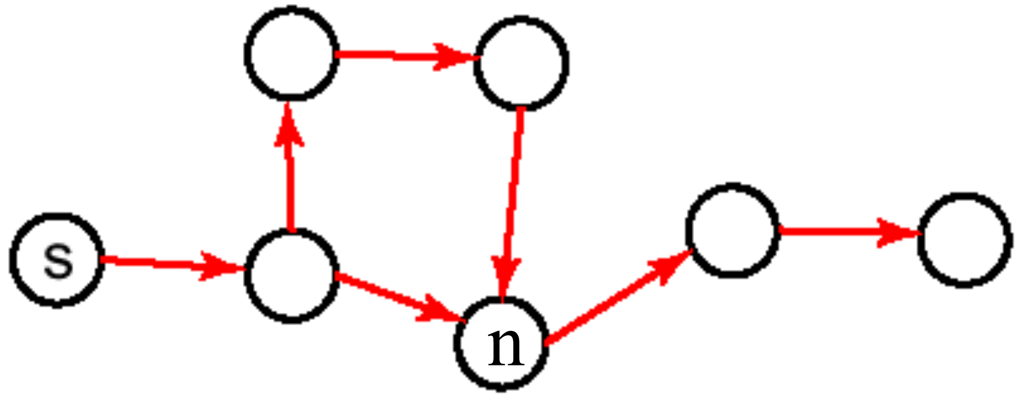
- With cycles or multiple parents, search tree can be exponential in the state space
  - E.g. state space with 2 actions from each state to next
  - With d + 1 states, search tree has depth d





- **2^d possible paths through the search space => exponentially larger search tree!**

Multiple Path Pruning

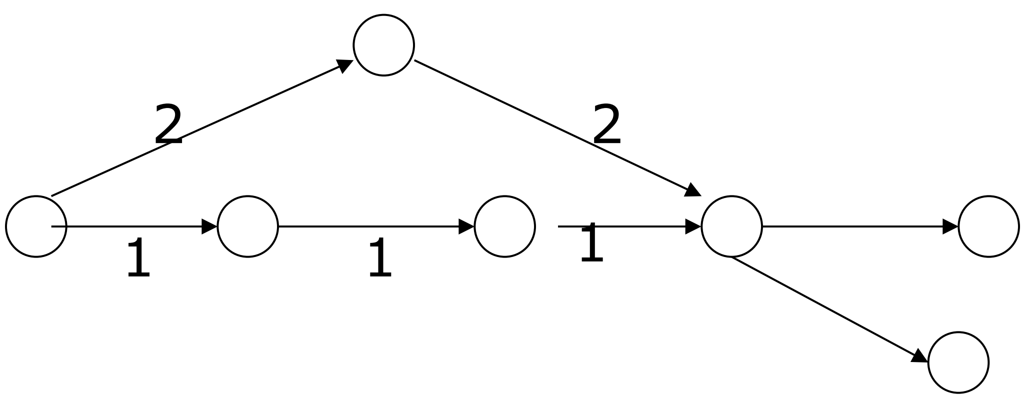


- If we only want one path to the solution
- Can prune path to a node *n* that has already been reached via a previous path
  - Store Closed := {all nodes n that have been expanded}
  - For newly expanded path p = $(n_1,\ldots,n_k,n)$
    - Check whether $n \in$ Closed
    - Subsumes cycle check
- Can implement by storing the path to each expanded node

Multiple-Path Pruning & Optimal Solutions

- Problem: what if a subsequent path to *n* is shorter than the first path to n, and we want an optimal solution ?

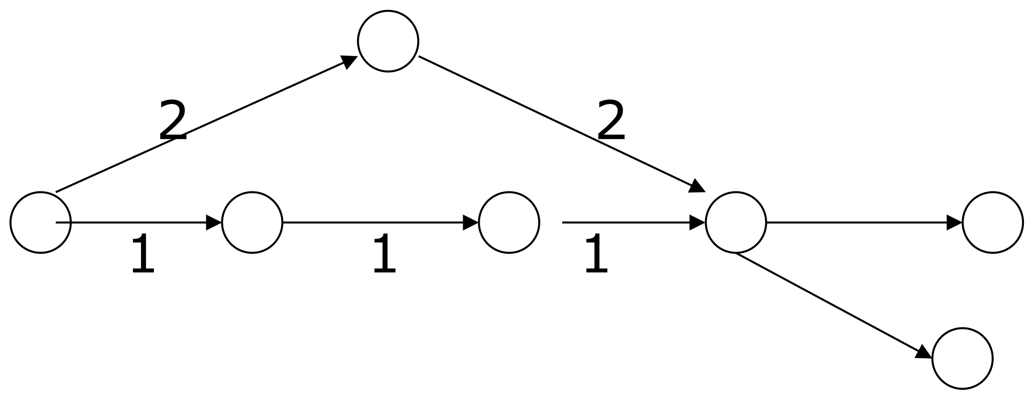
- Can remove all paths from the frontier that use the longer path: these can't be optimal.



Multiple-Path Pruning & Optimal Solutions

- Problem: what if a subsequent path to *n* is shorter than the first path to n, and we want just the optimal solution ?

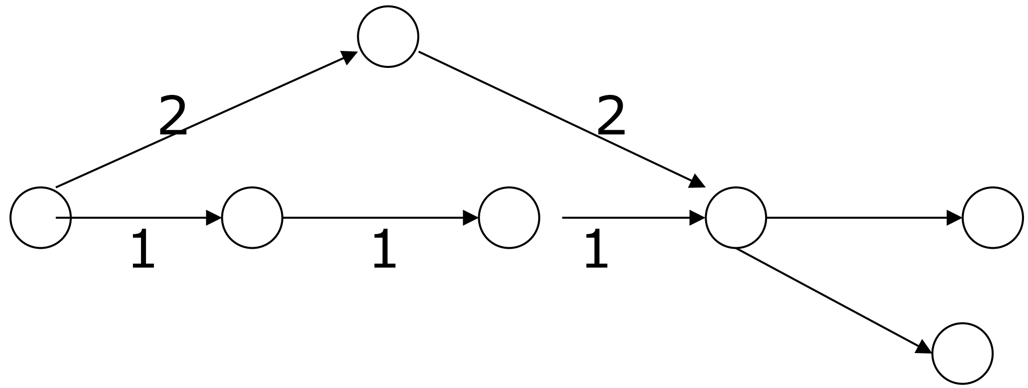
- Can change the initial segment of the paths on the frontier to use the shorter path



Multiple-Path Pruning & Optimal Solutions

- Problem: what if a subsequent path to *n* is shorter than the first path to n, and we want just the optimal solution ?

- Can prove that this can't happen for an algorithm?



- Which of the following algorithms always find the shortest path to nodes on the frontier first?

Least Cost Search First

A*

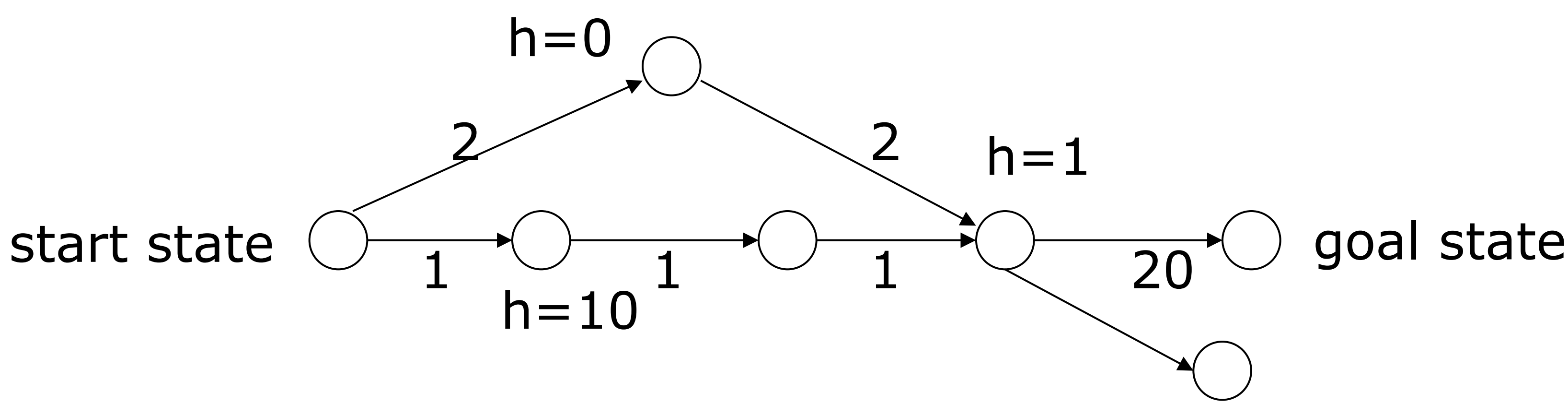
Both of the above

None of the above

- Which of the following algorithms always find the shortest path to nodes on the frontier first?
  - Only Least Cost First Search (like Dijkstra's algorithm)
  - For A* this is only guaranteed for nodes on the optimal solution path

Example: A* expands the upper path first

- Special conditions on the heuristic can recover the guarantee of LCFS for A*: the *monotone restriction* (See P&M text, Section 3.7.2)



Summary: pruning

- Sometimes we don't want pruning
  - Actually want multiple solutions (including non-optimal ones)

- Search tree can be exponentially larger than search space
  - So pruning is often important

- In DFS-type search algorithms
  - We can do cheap cycle checks: O(1)

- BFS-type search algorithms are memory-heavy already
  - We can store the path to each expanded node and do multiple path pruning

Lecture Overview

- Some clarifications & multiple path pruning
- Recap and more detail: Iterative Deepening and IDA*

Iterative Deepening DFS (short IDS): Motivation

Want low space complexity *and* completeness and optimality!

Key Idea: re-compute elements of the frontier
rather than saving them

	Complete	Optimal	Time	Space
DFS	N (Y if finite & no cycles)	N	$O(b^m)$	$O(mb)$
BFS	Y	Y	$O(b^m)$	$O(b^m)$
LCFS (when arc costs available)	Y Costs > 0	Y Costs ≥ 0	$\tilde{O}(b^m)$	$O(b^m)$
Best First (when *h* available)	N	N	$\tilde{O}(b^m)$	$O(b^m)$
A* (when arc costs and *h* available)	Y Costs > 0 *h* admissible	Y Costs ≥ 0 *h* admissible	$\tilde{O}(b^m)$	$O(b^m)$

Iterative Deepening DFS (IDS) in a Nutshell

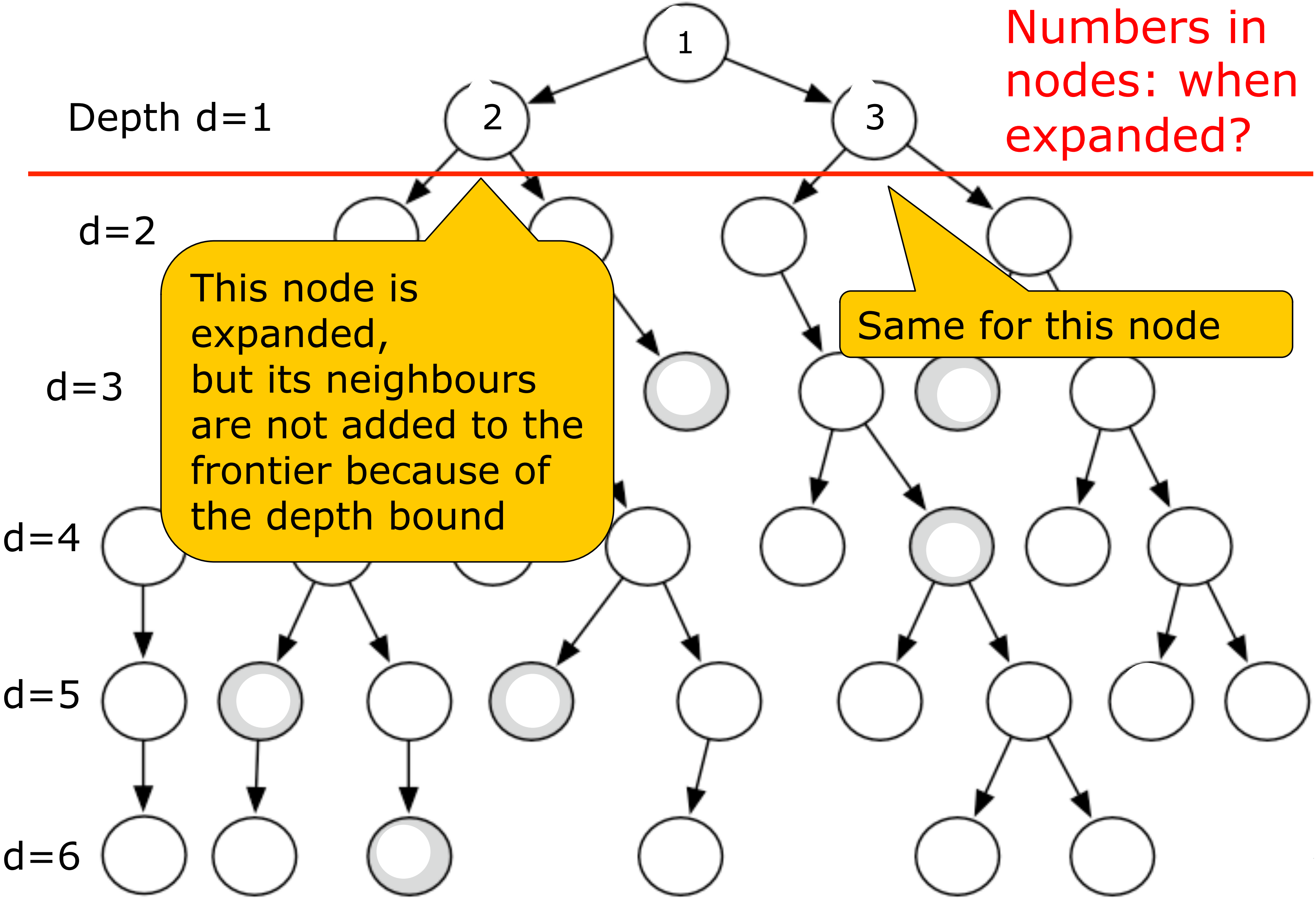
- Depth-bounded depth-first search: DFS on a leash
- For depth bound d, ignore any paths with longer length:
    - Not allowed to go too far away ⇒ backtrack ("fail unnaturally")
    - Only finite # paths with length ≤ d ⇒ terminates
- What is the memory requirement at depth bound d? (it is DFS!)
    - m=length of optimal solution path
    - b=branching factor

$O(b^m)$	$O(mb)$	$O(bd)$	$O(dm)$

- O(bd) ! It's a DFS, up to depth d.

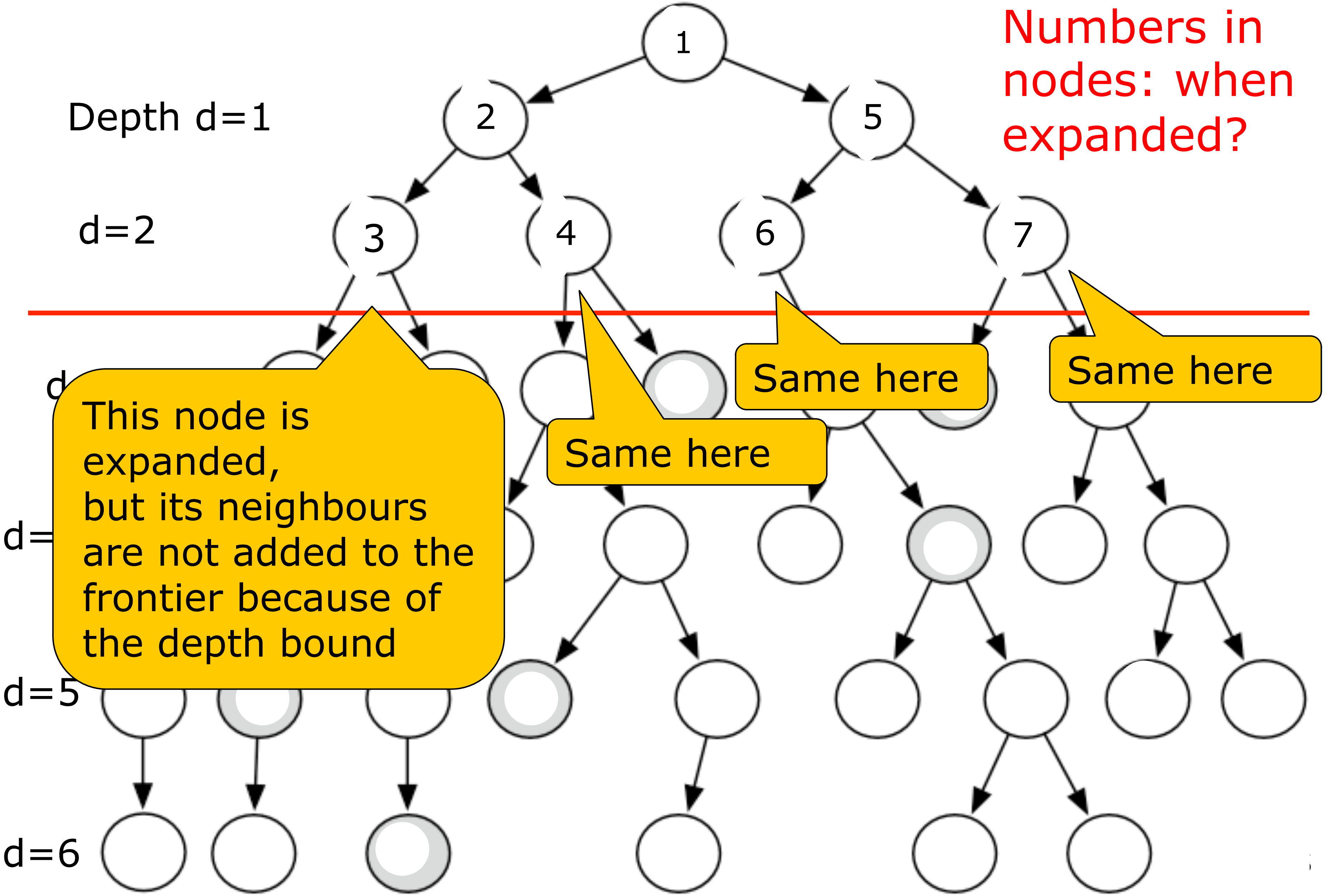
- Progressively increase the depth bound d
- Start at 1
- Then 2
- Then 3
- ...
- Until it finds the solution at depth m

Iterative Deepening DFS, depth bound = 1

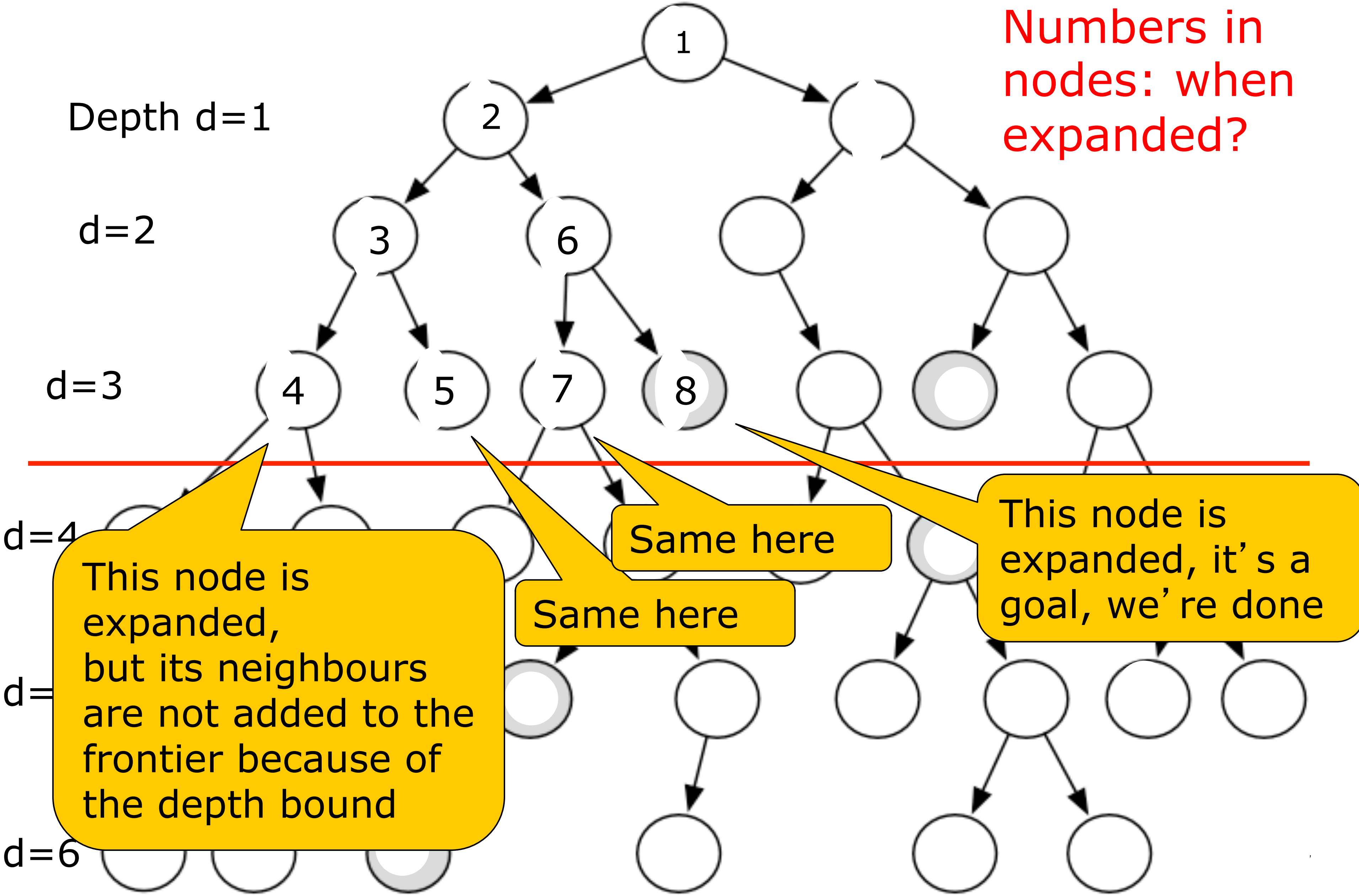


Iterative Deepening DFS, depth bound = 2

Numbers in nodes: when expanded?



Iterative Deepening DFS, depth bound = 3



Analysis of Iterative Deepening DFS (IDS)

- Space complexity

  $O(b^m)$ $O(m^b)$ $O(bm)$ $O(b+m)$

  – DFS scheme, only explore one branch at a time

- Complete? Yes No

  – Only finite # of paths up to depth m, doesn't explore longer paths

- Optimal? Yes No

  – Proof by contradiction

(Time) Complexity of IDS

The solution is at depth m, branching factor b

Total # of paths generated:

$$\leq b^m + (2\ b^{m-1}) + (3\ b^{m-2}) + ... + mb$$

We only exp d
paths at de m
once

We only expar
paths at depth 2
three times

We only expand
paths at depth m-1
twice

We expand paths at
depth 1 m times
(for every single
depth bound)

(Time) Complexity of IDS

From there on, it's just math:

Total # paths generated by IDS

$\leq b^m + (2\, b^{m-1}) + (3\, b^{m-2}) + ... + mb$

$= b^m\, (1\, b^0 + 2\, b^{-1} + 3\, b^{-2} + ... + m\, b^{1-m}\,)$

$$= b^m \left(\sum_{i=1}^{m} i b^{1-i}\right) = b^m \left(\sum_{i=1}^{m} i (b^{-1})^{i-1}\right)$$

If b > 1

$$\leq b^m \left(\sum_{i=0}^{\infty} i (b^{-1})^{i-1}\right) = b^m \left(\frac{1}{1-b^{-1}}\right)^2 = b^m \left(\frac{b}{b-1}\right)^2 \in O(b^m)$$

Geometric progression: for |r|<1: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$

$$\frac{d}{dr} \sum_{i=0}^{\infty} r^i = \sum_{i=0}^{\infty} i r^{i-1} = \frac{1}{(1-r)^2}$$

Conclusion for Iterative Deepening

- See the code in P&M, Section 3.7.3
- Even though it re-does what seems like a lot of work
  - Actually, compared to how much work there is at greater depths, it's not a lot of work
  - Re-does the first levels most often
    - But those are the cheapest ones
- Time Complexity $O(b^m)$
  - Just like a single DFS
  - Just like the last depth-bounded DFS
    - That last depth bounded DFS dominates the search complexity
- Space complexity: $O(bm)$
- Optimal
- Complete

(Heuristic) Iterative Deepening: IDA*

- Like Iterative Deepening DFS
  - But the "depth" bound is measured in terms of the f value
  - f-value-bounded DFS: DFS on a f-value leash
  - IDA* is a bit of a misnomer
    - The only thing it has in common with A* is that it uses the f value $f(p) = cost(p) + h(p)$
    - It does NOT expand the path with lowest f value. It is doing DFS!
    - But f-value-bounded DFS doesn't sound as good …
- Start with f-value = f(s) (s is start node)
- If you don't find a solution at a given f-value
  - Increase the bound:
    to the minimum of the f-values that exceeded the previous bound
- Will explore all nodes with f value < f_{min} (optimal one)

Analysis of Iterative Deepening A* (IDA*)

- Complete and optimal? Same conditions as A*
  - h is admissible
  - all arc costs > 0
  - finite branching factor

- Time complexity: $O(b^m)$
  - Same argument as for Iterative Deepening DFS

- Space complexity:

  $O(b^m)$ $O(m^b)$ $O(bm)$ $O(b+m)$

  - Same argument as for Iterative Deepening DFS

Search methods so far

	Complete	Optimal	Time	Space
DFS	N (Y if no cycles)	N	$O(b^m)$	$O(mb)$
BFS	Y	Y	$O(b^m)$	$O(b^m)$
IDS	Y	Y	$O(b^m)$	$O(mb)$
LCFS (when arc costs available)	Y Costs > 0	Y Costs >=0	$\tilde{O}(b^m)$	$O(b^m)$
Best First (when *h* available)	N	N	$\tilde{O}(b^m)$	$O(b^m)$
A* (when arc costs and *h* available)	Y Costs > 0 *h* admissible	Y Costs >=0 *h* admissible	$\tilde{O}(b^m)$	$O(b^m)$
IDA*	Y (same cond. as A*)	Y	$O(b^m)$	$O(mb)$

Learning Goals for today’s class

- Define/read/write/trace/debug different search algorithms
  - In more detail today: Iterative Deepening, Iterative Deepening A*
- Apply basic properties of search algorithms:
  - completeness, optimality, time and space complexity

Announcements:

- Practice exercises are out on home page.
  - Heuristic search
  - Please use them! (Only takes 5 min. if you understood things…)
- Assignment 1 is out: see Connect

Learning Goals for search

- **Identify** real world examples that make use of deterministic, goal-driven search agents
- **Assess** the size of the search space of a given search problem.
- **Implement** the generic solution to a search problem.
- **Apply** basic properties of search algorithms:
  - completeness, optimality, time and space complexity
- **Select** the most appropriate search algorithms for specific problems.
- **Define/read/write/trace/debug** different search algorithms
- **Construct** heuristic functions for specific search problems
- **Formally prove** A* optimality.
- **Define optimally** efficient

Coming up: Constraint Satisfaction Problems

- Read chapter 4

- Get busy with assignment 1