4 Maximum likelihood

4.1

The likelihood of the set of iid observations is

$$p(x_{1:N}|\theta) = \prod_{i=1}^{N} \frac{1}{\theta}\mathbb{I}(x_i < \theta) \tag{1}$$

We ignore the lower bound on x_i for notational simplicity, it does not affect our selection of θ.

For the following, we use $\mathbb{I}(a < \theta) \times \mathbb{I}(b < \theta) = \mathbb{I}(\max\{a, b\} < \theta)$.

$$\begin{aligned}
\hat{\theta}_{MLE} &= \operatorname*{argmax}_{\theta} p(x_{1:N}|\theta) && (2)\\
&= \operatorname*{argmax}_{\theta} \prod_{i=1}^{N} \frac{1}{\theta}\mathbb{I}(x_i < \theta) && (3)\\
&= \operatorname*{argmax}_{\theta} \frac{1}{\theta^N}\mathbb{I}(\max x_{1:N} < \theta) && (4)\\
&= \max x_{1:N} && (5)
\end{aligned}$$

If $\hat{\theta}_{MLE}$ were any lower than $\max x_{1:N}$, then there would be an x_i with a zero likelihood. Any greater, and the $\frac{1}{\theta^N}$ term would decrease, giving a lower likelihood.

4.2

See figures at end of document.

4.3

$$\begin{aligned}
p(\theta|x_{1:N}) &= \frac{p(x_{1:N}|\theta)p(\theta)}{p(x_{1:N})} && (6)\\
&\propto p(x_{1:N}|\theta)p(\theta) && (7)\\
&\propto \left[\frac{1}{\theta^N}\mathbb{I}(\max x_{1:N} < \theta)\right] \times \left[\alpha\beta^{\alpha}\theta^{-\alpha-1}\mathbb{I}(\beta < \theta)\right] && (8)\\
&\propto \left[\frac{1}{\theta^N}\mathbb{I}(\max x_{1:N} < \theta)\right] \times \left[\theta^{-\alpha-1}\mathbb{I}(\beta < \theta)\right] && (9)\\
&\propto \theta^{-N-\alpha-1}\mathbb{I}(\max\{\beta, x_{1:N}\} < \theta) && (10)
\end{aligned}$$

This has the same form as the Pareto distribution, with updated parameters:

$$\begin{aligned}
\alpha' &= \alpha + N && (11)\\
\beta' &= \max\{\beta, x_{1:N}\} && (12)
\end{aligned}$$

4.4

$$\begin{aligned}
\hat{\theta}_{MAP} &= \underset{\theta}{\text{argmax}}\, \text{Pareto}(x|\alpha', \beta') && (13)\\
&= \underset{\theta}{\text{argmax}}\, \alpha' \beta'^{\alpha'} \theta^{-\alpha'-1} \mathbb{I}(\beta' < \theta) && (14)\\
&= \underset{\theta}{\text{argmax}}\, \theta^{-\alpha'-1} \mathbb{I}(\beta' < \theta) && (15)\\
&= \beta' && (16)
\end{aligned}$$

The posterior is of the same form as the expression for the likelihood in question 4.1, therefore we can use the same argument for finding the maximum.

4.5

Not required.

4.6

Here we use the expressions we found in question 4.3 for the posterior parameters. $N = 3$ and $\max x_{1:N} = 1.9$.

4.6.1

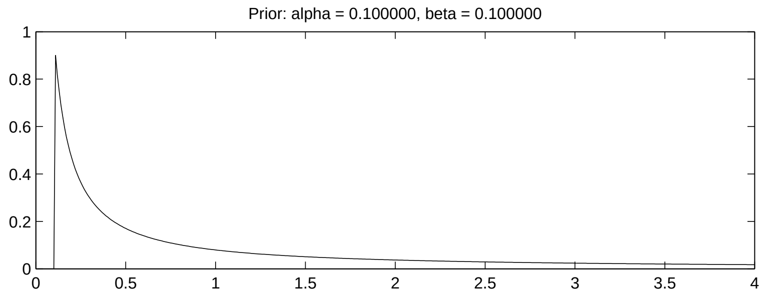
$$(\alpha, \beta) = (0.1, 0.1) \quad (17)$$
$$(\alpha', \beta') = (3.1, 1.9) \quad (18)$$

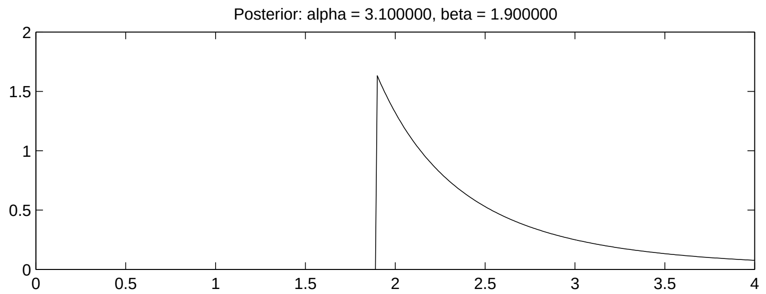
4.6.2

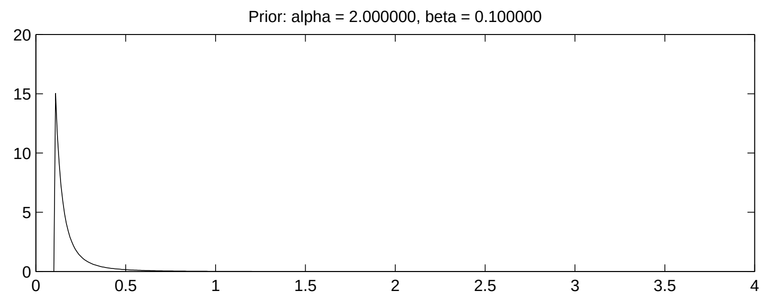
$$(\alpha, \beta) = (2.0, 0.1) \quad (19)$$
$$(\alpha', \beta') = (5.0, 1.9) \quad (20)$$

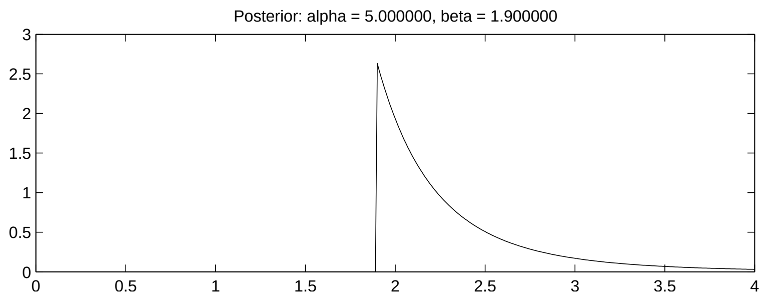
4.6.3

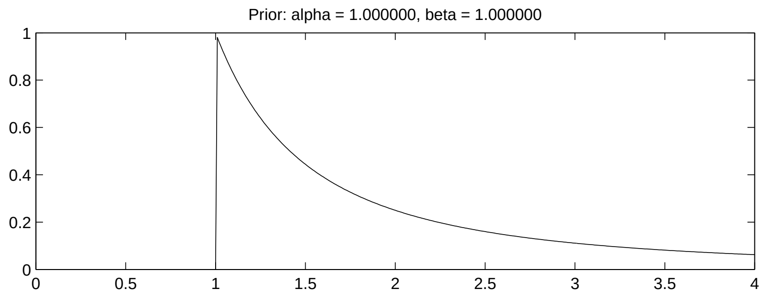
$$(\alpha, \beta) = (1.0, 0.1) \quad (21)$$
$$(\alpha', \beta') = (4.0, 1.9) \quad (22)$$

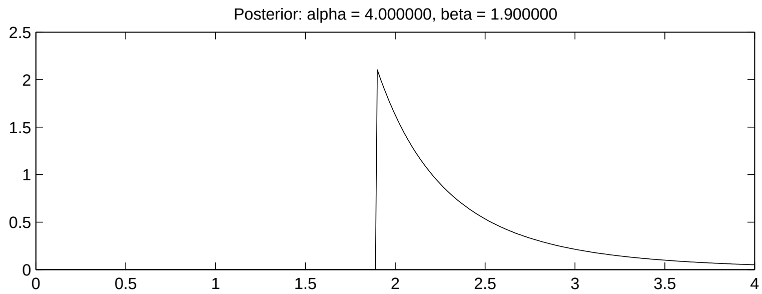

Prior: alpha = 0.100000, beta = 0.100000
0 0.2 0.4 0.6 0.8 1
0 0.5 1 1.5 2 2.5 3 3.5 4


Posterior: alpha = 3.100000, beta = 1.900000
0 0.5 1 1.5 2
0 0.5 1 1.5 2 2.5 3 3.5 4


Prior: alpha = 2.000000, beta = 0.100000
0 5 10 15 20
0 0.5 1 1.5 2 2.5 3 3.5 4


Posterior: alpha = 5.000000, beta = 1.900000
0 0.5 1 1.5 2 2.5 3
0 0.5 1 1.5 2 2.5 3 3.5 4


Prior: alpha = 1.000000, beta = 1.000000
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4


Posterior: alpha = 4.000000, beta = 1.900000
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4