Q2 - 3

Suppose that $\mathbf{v}_i$ is an eigenvector of $\mathbf{X^TX}$.

$$\begin{aligned}
\mathbf{X^TXv_i} &= \lambda_i \mathbf{v_i} \\
\implies \mathbf{XX^TXv_i} &= \lambda_i \mathbf{Xv_i} \\
\implies \frac{1}{N}\mathbf{XX^TXv_i} &= \frac{\lambda_i}{N}\mathbf{Xv_i} \\
\implies (\frac{1}{N}\mathbf{XX^T})(\frac{1}{\sqrt{\lambda_i}}\mathbf{Xv_i}) &= \frac{\lambda_i}{N}(\frac{1}{\sqrt{\lambda_i}}\mathbf{Xv_i}) \\
\implies \mathbf{\Sigma u_i} &= \frac{\lambda_i}{N}\mathbf{u_i}
\end{aligned}$$

So $\mathbf{v}_i$ is an eigenvector of Σ, with associated eigenvalue of $\frac{\lambda_i}{N}$

Now assume that $||\mathbf{v_i}||_2 = 1$, we then get:

$$\begin{aligned}
||\mathbf{u_i}||_2^2 &= \mathbf{u_i^T u_i} \\
&= (\frac{1}{\sqrt{\lambda_i}}\mathbf{Xv_i})^\mathbf{T}(\frac{1}{\sqrt{\lambda_i}}\mathbf{Xv_i}) \\
&= \frac{1}{\lambda_i}\mathbf{v_i^T X^T X v_i} \\
&= \frac{1}{\lambda_i}\mathbf{v_i^T}\lambda_i \mathbf{v_i} \\
&= ||\mathbf{v_i}||_2^2 \\
&= 1
\end{aligned}$$